Today’s Outline - February 13, 2018
• Ideal refractive surface
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- Fresnel lenses and zone plates
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- Research papers on refraction
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• Kinematical diffraction
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Homework Assignment #02:
Problems on Blackboard
due Tuesday, February 13, 2018
Refractive optics

Just as with visible light, it is possible to make refractive optics for x-rays.

\[ n \approx 1 - \delta, \quad \delta \approx 10^{-5} \]

\[ f \approx 100 \text{m}! \]

x-ray lenses are complementary to those for visible light. Getting manageable focal distances requires making compound lenses.
Refractive optics

Just as with visible, light, it is possible to make refractive optics for x-rays:

visible light:

\[ n \sim 1.2 - 1.5 \]
\[ f \sim 0.1m \]
Refractive optics

Just as with visible, light, it is possible to make refractive optics for x-rays:

Visible light:

\[ n \sim 1.2 - 1.5 \]
\[ f \sim 0.1m \]

X-rays:

\[ n \approx 1 - \delta, \ \delta \sim 10^{-5} \]
\[ f \sim 100m! \]
Refractive optics

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Visible light:

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\[ f \sim 0.1 \text{m} \]

X-rays:

\[ n \approx 1 - \delta, \quad \delta \sim 10^{-5} \]
\[ f \sim 100 \text{m!} \]

X-ray lenses are complementary to those for visible light.
Refractive optics

Just as with visible, light, it is possible to make refractive optics for x-rays:

visible light:
\[
\begin{align*}
n & \sim 1.2 - 1.5 \\
f & \sim 0.1 \text{m}
\end{align*}
\]

x-rays:
\[
\begin{align*}
n & \approx 1 - \delta, \ \delta \sim 10^{-5} \\
f & \sim 100 \text{m!}
\end{align*}
\]

x-ray lenses are complementary to those for visible light getting manageable focal distances requires making compound lenses.
Focal length of a compound lens

Start with a 3-element compound lens, calculate effective focal length.
Focal length of a compound lens

Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f}$$
Focal length of a compound lens

Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$

\[
\frac{1}{i} + \frac{1}{o} = \frac{1}{f} \rightarrow \frac{1}{i} = \frac{1}{f} - \frac{1}{o}
\]
Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$.

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f} \Rightarrow \frac{1}{i} = \frac{1}{f} - \frac{1}{o}$$

$$\frac{1}{i_1} = \frac{1}{f_1} - \frac{1}{o_1}$$
Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$

\[ f_1 = f, \quad o_1 = \infty \]
Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, \( f \)

\[
\begin{align*}
\frac{1}{i} + \frac{1}{o} &= \frac{1}{f} \rightarrow \frac{1}{i} &= \frac{1}{f} - \frac{1}{o} \\
\frac{1}{i_1} &= \frac{1}{f_1} - \frac{1}{o_1} \rightarrow \frac{1}{i_1} &= \frac{1}{f} \rightarrow i_1 = f
\end{align*}
\]

\( f_1 = f, \ o_1 = \infty \)
Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$

$$f_1 = f, \ o_1 = \infty$$

for the second lens, the image $i_1$ is a virtual object, $o_2 = -i_1$
Focal length of a compound lens

Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$

1. For the first lens, $f_1 = f$, $o_1 = \infty$

2. For the second lens, the image $i_1$ is a virtual object, $o_2 = -i_1$
Focal length of a compound lens

Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$

\[ \frac{1}{i_1} = \frac{1}{f_1} - \frac{1}{o_1} \rightarrow \frac{1}{i_1} = \frac{1}{f} \rightarrow i_1 = f \]

\[ \frac{1}{i_2} = \frac{1}{f_2} - \frac{1}{o_2} \rightarrow \frac{1}{i_2} = \frac{1}{f} + \frac{1}{f} \rightarrow i_2 = \frac{f}{2} \]

for the second lens, the image $i_1$ is a virtual object, $o_2 = -i_1$
Focal length of a compound lens

Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, $f$

$$f_1 = f, \quad o_1 = \infty$$

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similarly for the third lens, $o_3 = -i_2$
Focal length of a compound lens

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Focal length of a compound lens

Start with a 3-element compound lens, calculate effective focal length assuming each lens has the same focal length, \( f \)

\[
\frac{1}{i} + \frac{1}{o} = \frac{1}{f} \rightarrow \frac{1}{i} = \frac{1}{f} - \frac{1}{o}
\]

\[
\frac{1}{i_1} = \frac{1}{f_1} - \frac{1}{o_1} \rightarrow \frac{1}{i_1} = \frac{1}{f} \rightarrow i_1 = f
\]

\[
\frac{1}{i_2} = \frac{1}{f_2} - \frac{1}{o_2} \rightarrow \frac{1}{i_2} = \frac{1}{f} + \frac{1}{f} \rightarrow i_2 = \frac{f}{2}
\]

\[
\frac{1}{i_2} = \frac{1}{f_2} - \frac{1}{o_2} \rightarrow \frac{1}{i_2} = \frac{1}{f} + \frac{2}{f} \rightarrow i_2 = \frac{f}{3}
\]

so for \( N \) lenses \( f_{\text{eff}} = f/N \)
Rephasing distance

A spherical surface is not the ideal lens as it introduces aberrations. Derive the ideal shape for perfect focusing of x-rays.
Rephasing distance

A spherical surface is not the ideal lens as it introduces aberrations. Derive the ideal shape for perfect focusing of x-rays.

consider two waves, one traveling inside the solid and the other in vacuum,

\[ \lambda = \lambda_0 / (1 - \delta) \approx \lambda_0 (1 + \delta) \]
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\( \lambda_0 \) \( \lambda_0 (1+\delta) \)
Rephasing distance

A spherical surface is not the ideal lens as it introduces aberrations. Derive the ideal shape for perfect focusing of x-rays.

Consider two waves, one traveling inside the solid and the other in vacuum.

\[ \lambda = \frac{\lambda_0}{1 - \delta} \approx \lambda_0 (1 + \delta) \]

If the two waves start in phase, they will be in phase once again after a distance

\[ \Lambda = (N + 1)\lambda_0 = N\lambda_0 (1 + \delta) \]
Rephasing distance

A spherical surface is not the ideal lens as it introduces aberrations. Derive the ideal shape for perfect focusing of x-rays.

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\[
\Lambda = (N + 1)\lambda_0 = N\lambda_0 (1 + \delta)
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\[
N\lambda_0 + \lambda_0 = N\lambda_0 + N\delta\lambda_0
\]
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\Lambda = (N + 1)\lambda_0 = N\lambda_0(1 + \delta)
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N\lambda_0 + \lambda_0 = N\lambda_0 + N\delta\lambda_0 \quad \rightarrow \quad \lambda_0 = N\delta\lambda_0 \quad \rightarrow \quad N = \frac{1}{\delta}
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Rephasing distance

A spherical surface is not the ideal lens as it introduces aberrations. Derive the ideal shape for perfect focusing of x-rays.

Consider two waves, one traveling inside the solid and the other in vacuum, $\lambda = \lambda_0/(1 - \delta) \approx \lambda_0(1 + \delta)$.

If the two waves start in phase, they will be in phase once again after a distance $\Lambda = (N + 1)\lambda_0 = N\lambda_0(1 + \delta)$.

The phase difference is given by $N\lambda_0 + \lambda_0 = N\lambda_0 + N\delta\lambda_0 \implies \lambda_0 = N\delta\lambda_0 \implies N = \frac{1}{\delta}$.

The distance is $\Lambda = N\lambda_0 = \frac{\lambda_0}{\delta}$. 
Rephasing distance

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\[ \lambda = \frac{\lambda_0}{1 - \delta} \approx \lambda_0 (1 + \delta) \]

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\[ \Lambda = (N + 1) \lambda_0 = N \lambda_0 (1 + \delta) \]

\[ N \lambda_0 + \lambda_0 = N \lambda_0 + N \delta \lambda_0 \quad \rightarrow \quad \lambda_0 = N \delta \lambda_0 \quad \rightarrow \quad N = \frac{1}{\delta} \]

\[ \Lambda = N \lambda_0 = \frac{\lambda_0}{\delta} = \frac{2\pi}{\lambda_0 r_0 \rho} \]
Rephasing distance

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\[ \Lambda = (N + 1)\lambda_0 = N\lambda_0 (1 + \delta) \]

\[ N\lambda_0 + \lambda_0 = N\lambda_0 + N\delta\lambda_0 \quad \rightarrow \quad \lambda_0 = N\delta\lambda_0 \quad \rightarrow \quad N = \frac{1}{\delta} \]

\[ \Lambda = N\lambda_0 = \frac{\lambda_0}{\delta} = \frac{2\pi}{\lambda_0 r_0 \rho} \approx 10 \mu m \]
Ideal interface profile

The wave exits the material into vacuum through a surface of profile \( h(x) \), and is twisted by an angle \( \alpha \).

Follow the path of two points on the wave-front, \( A \) and \( A' \), as they propagate to \( B \) and \( B' \).

From the \( \Delta AA'B \) triangle and from the \( \Delta BCB' \) triangle using \( \Lambda = \frac{\lambda_0}{\delta} \frac{h'(x)}{\Delta x} \approx \lambda_0 h'(x) \delta \approx \lambda_0 \delta \Delta x = h'(x) \lambda_0 \Lambda \).
The wave exits the material into vacuum through a surface of profile \( h(x) \), and is twisted by an angle \( \alpha \).
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The wave exits the material into vacuum through a surface of profile $h(x)$, and is twisted by an angle $\alpha$. Follow the path of two points on the wave-front, $A$ and $A'$ as they propagate to $B$ and $B'$. From the $AA'B'$ triangle

$$\lambda_0 (1 + \delta) = h'(x) \Delta x$$
The wave exits the material into vacuum through a surface of profile $h(x)$, and is twisted by an angle $\alpha$. Follow the path of two points on the wave-front, $A$ and $A'$ as they propagate to $B$ and $B'$. 

from the $AA'B'$ triangle

$$\lambda_0 (1 + \delta) = h'(x) \Delta x \quad \rightarrow \quad \Delta x \approx \frac{\lambda_0}{h'(x)}$$
The wave exits the material into vacuum through a surface of profile $h(x)$, and is twisted by an angle $\alpha$. Follow the path of two points on the wave-front, $A$ and $A'$ as they propagate to $B$ and $B'$.

From the $AA'B'$ triangle

and from the $BCB'$ triangle

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From the \( AA'B' \) triangle

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\[
\lambda_0 (1 + \delta) = h'(x) \Delta x \quad \rightarrow \quad \Delta x \approx \frac{\lambda_0}{h'(x)}
\]

\[
\alpha(x) \approx \frac{\lambda_0 \delta}{\Delta x}
\]
The wave exits the material into vacuum through a surface of profile $h(x)$, and is twisted by an angle $\alpha$. Follow the path of two points on the wavefront, $A$ and $A'$ as they propagate to $B$ and $B'$.

from the $AA'B'$ triangle
and from the $BCB'$ triangle

$$\lambda_0 (1 + \delta) = h'(x) \Delta x \quad \rightarrow \quad \Delta x \approx \frac{\lambda_0}{h'(x)}$$

$$\alpha(x) \approx \frac{\lambda_0 \delta}{\Delta x} = h'(x) \delta$$
The wave exits the material into vacuum through a surface of profile $h(x)$, and is twisted by an angle $\alpha$. Follow the path of two points on the wave-front, $A$ and $A'$ as they propagate to $B$ and $B'$.

from the $AA'B'$ triangle
and from the $BCB'$ triangle
using $\Lambda = \lambda_0/\delta$

$$\lambda_0 (1 + \delta) = h'(x) \Delta x \quad \rightarrow \quad \Delta x \approx \frac{\lambda_0}{h'(x)}$$

$$\alpha(x) \approx \frac{\lambda_0 \delta}{\Delta x} = h'(x) \delta$$
Ideal interface profile

The wave exits the material into vacuum through a surface of profile $h(x)$, and is twisted by an angle $\alpha$. Follow the path of two points on the wavefront, $A$ and $A'$ as they propagate to $B$ and $B'$.

from the $AA'B'$ triangle
and from the $BCB'$ triangle
using $\Lambda = \lambda_0 / \delta$

$$
\lambda_0 (1 + \delta) = h'(x) \Delta x \quad \rightarrow \quad \Delta x \approx \frac{\lambda_0}{h'(x)}
$$

$$
\alpha(x) \approx \frac{\lambda_0 \delta}{\Delta x} = h'(x) \delta = h'(x) \frac{\lambda_0}{\Lambda}
$$
Ideal interface profile

If the desired focal length of this lens is $f$, the wave must be redirected at an angle which depends on the distance from the optical axis.

\[ \alpha(x) = \frac{x}{f} \]

This can be directly integrated:

\[ \Lambda = \frac{x^2}{2f\lambda_0} = \left[ \frac{x}{\sqrt{2f\lambda_0}} \right]^2 \]

A parabola is the ideal surface shape for focusing by refraction.
If the desired focal length of this lens is $f$, the wave must be redirected at an angle which depends on the distance from the optical axis

$$\alpha(x) = \frac{x}{f}$$
Ideal interface profile

If the desired focal length of this lens is $f$, the wave must be redirected at an angle which depends on the distance from the optical axis:

$$\alpha(x) = \frac{x}{f}$$

Combining, we have

$$\frac{\lambda_0 h'(x)}{\Lambda} = \frac{x}{f}$$
Ideal interface profile

If the desired focal length of this lens is $f$, the wave must be redirected at an angle which depends on the distance from the optical axis

$$\alpha(x) = \frac{x}{f}$$

combining, we have

$$\frac{\lambda_0 h'(x)}{\Lambda} = \frac{x}{f} \Rightarrow \frac{h'(x)}{\Lambda} = \frac{x}{f \lambda_0}$$
If the desired focal length of this lens is $f$, the wave must be redirected at an angle which depends on the distance from the optical axis

$$\alpha(x) = \frac{x}{f}$$

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$$\frac{\lambda_0 h'(x)}{\Lambda} = \frac{x}{f} \quad \rightarrow \quad \frac{h'(x)}{\Lambda} = \frac{x}{f\lambda_0}$$

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Ideal interface profile

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$$\frac{h(x)}{\Lambda} = \frac{x^2}{2f \lambda_0}$$
Ideal interface profile

If the desired focal length of this lens is $f$, the wave must be redirected at an angle which depends on the distance from the optical axis

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$$\frac{h(x)}{\Lambda} = \frac{x^2}{2f\lambda_0} = \left[ \frac{x}{\sqrt{2f\lambda_0}} \right]^2$$
Ideal interface profile

If the desired focal length of this lens is $f$, the wave must be redirected at an angle which depends on the distance from the optical axis

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$$\frac{h(x)}{\Lambda} = \frac{x^2}{2f \lambda_0} = \left[ \frac{x}{\sqrt{2f \lambda_0}} \right]^2$$

a parabola is the ideal surface shape for focusing by refraction
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

\[
f = \frac{x^2 \Lambda}{2 \lambda_0 h(x)}
\]
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

\[ f = \frac{x^2 \Lambda}{2 \lambda_0 h(x)} = \frac{1}{2 \delta} \frac{x^2}{h(x)} \]
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

\[ f = \frac{x^2 \Lambda}{2 \lambda_0 h(x)} = \frac{1}{2 \delta} \frac{x^2}{h(x)} \]

or alternatively

\[ f = \frac{1}{\delta} \frac{x}{h'(x)} \]
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From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

\[ f = \frac{x^2 \Lambda}{2 \lambda_0 h(x)} = \frac{1}{2\delta} \frac{x^2}{h(x)} \quad \text{or alternatively} \quad f = \frac{1}{\delta} \frac{x}{h'(x)} \]

If the surface is a circle instead of a parabola

\[ h(x) = \sqrt{R^2 - x^2} \]

and for \( x \ll R \)

\[ h(x) \approx R (1 - \frac{x^2}{2R^2}) \]

so we have

\[ f \approx \frac{R^2}{2} \delta \]

for \( N \) circular lenses

\[ f_n \approx \frac{R^2}{2} N \delta \]
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

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If the surface is a circle instead of a parabola

\[ h(x) = \sqrt{R^2 - x^2} \quad \text{and for } x \ll R \quad h(x) \approx R \left(1 - \frac{1}{2} \frac{x^2}{R^2}\right) \]
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

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f = \frac{x^2 \Lambda}{2 \lambda_0 h(x)} = \frac{1}{2 \delta} \frac{x^2}{h(x)}
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or alternatively

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f = \frac{1}{\delta} \frac{x}{h'(x)}
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If the surface is a circle instead of a parabola

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h(x) = \sqrt{R^2 - x^2} \quad \text{and for } x \ll R \quad h(x) \approx R \left(1 - \frac{1}{2} \frac{x^2}{R^2}\right)
\]

\[
x^2 = R^2 - h^2(x) \approx R^2
\]
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

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\[ x^2 = R^2 - h^2(x) \approx R^2 \quad \text{so we have} \quad f \approx \frac{R}{2\delta} \]
Focal length of circular lens

From the previous expression for the ideal parabolic surface, the focal length can be written in terms of the surface profile.

\[ f = \frac{x^2 \Lambda}{2 \lambda_0 h(x)} = \frac{1}{2\delta} \frac{x^2}{h(x)} \quad \text{or alternatively} \quad f = \frac{1}{\delta} \frac{x}{h'(x)} \]

If the surface is a circle instead of a parabola

\[ h(x) = \sqrt{R^2 - x^2} \quad \text{and for} \quad x \ll R \quad h(x) \approx R \left(1 - \frac{1}{2} \frac{x^2}{R^2} \right) \]

\[ x^2 = R^2 - h^2(x) \approx R^2 \quad \text{so we have} \quad f \approx \frac{R}{2\delta} \]

for \( N \) circular lenses

\[ f_n \approx \frac{R}{2N\delta} \]
Focussing by a beryllium lens

For 50 holes of radius $R = 1\text{mm}$ in beryllium (Be) at $E = 10\text{keV}$, we can calculate the focal length, knowing $\delta = 3.41 \times 10^{-6}$.

$$f = \frac{R^2}{N \delta} = \frac{1 \times 10^{-3} \text{m}^2}{50 \times (3.41 \times 10^{-6})} = 2.93 \text{m}$$

depending on the wall thickness of the lenslets, the transmission can be up to 74%.

Focussing by a beryllium lens

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$$f_N = \frac{R}{2N\delta}$$

Focussing by a beryllium lens

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$$f_N = \frac{R}{2N\delta} = \frac{1 \times 10^{-3}\text{m}}{2(50)(3.41 \times 10^{-6})}$$

For 50 holes of radius $R = 1\text{mm}$ in beryllium (Be) at $E = 10\text{keV}$, we can calculate the focal length, knowing $\delta = 3.41 \times 10^{-6}$

$$f_N = \frac{R}{2N\delta} = \frac{1 \times 10^{-3}\text{m}}{2(50)(3.41 \times 10^{-6})} = 2.93\text{m}$$

For 50 holes of radius $R = 1\text{mm}$ in beryllium (Be) at $E = 10\text{keV}$, we can calculate the focal length, knowing $\delta = 3.41 \times 10^{-6}$

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depending on the wall thickness of the lenslets, the transmission can be up to 74%.

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This design has also been used to make lenses out of lithium metal.

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Cut diagonally to expose variable number of “lenses” to a horizontal beam

Horizontal translation allows change in focal length but it is quantized, not continuous

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\[ \text{Start with a 2 hole CRL. Rotate by an angle } \chi \text{ about vertical axis giving an effective change in the number of "lenses" by a factor } \frac{1}{\cos \chi}. \]

At \( E = 5.5 \text{ keV} \) and \( \chi = 0^\circ \), height is over 120 \( \mu \text{m} \). At \( \chi = 30^\circ \), it is under 50 \( \mu \text{m} \). Optimal focus is 20 \( \mu \text{m} \) at \( \chi = 40^\circ \).

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How to make a Fresnel lens

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aspect ratio too large for a stable structure and absorption would be too large!
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Each block of thickness $\Lambda$ serves no purpose for refraction but only attenuates the wave.

This material can be removed and the remaining material collapsed to produce a Fresnel lens which has the same optical properties as the parabolic lens as long as $f \gg N\Lambda$ where $N$ is the number of zones.
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Fresnel lens dimensions

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The diameter of the entire lens is thus $2\xi_N = 2\sqrt{N} = \frac{\Delta\xi_N}{\sqrt{N}}$.
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Fresnel lens example

In terms of the unscaled variables

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Making a Fresnel zone plate

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In practice, since the outermost zones are very small, zone plates are generally fabricated as alternating zones (rings for 2D) of materials with a large Z-contrast, such as Au/Si or W/C.
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In practice, since the outermost zones are very small, zone plates are generally fabricated as alternating zones (rings for 2D) of materials with a large Z-contrast, such as Au/Si or W/C.

This kind of zone plate is not as efficient as a true Fresnel lens would be in the x-ray regime. Nevertheless, efficiencies up to 35% have been achieved.
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Detail view of outer zones

Scattering from two electrons

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\[ \vec{Q} = (\vec{k} - \vec{k}') \]

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The scattering from the second electron will have a phase shift of \( \phi = \vec{Q} \cdot \vec{r} \).

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Since experiments measure $I \propto A^2$, the phase information is lost. This is a problem if we don’t know the specific orientation of the scattering system relative to the x-ray beam.
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We will now look at the consequences of this orientation and generalize to more than two electrons.
Two electrons — fixed orientation

The expression

\[ I(\vec{Q}) = 2r_0^2 \left( 1 + \cos(\vec{Q} \cdot \vec{r}) \right) \]

assumes that the two electrons have a specific, fixed orientation. In this case the intensity as a function of \( Q \) is.
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Fixed orientation is not the usual case, particularly for solution and small-angle scattering.
Orientation averaging

Consider scattering from two arbitrary electron distributions, $f_1$ and $f_2$. $A(\vec{Q})$, is given by
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\[ \left\langle e^{i\vec{Q} \cdot \vec{r}} \right\rangle = \frac{\int e^{iQr \cos \theta \sin \theta} \cos \theta \sin \theta d\theta d\phi}{\int \sin \theta d\theta d\phi} \]

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\]

\[
= \frac{2\pi}{4\pi} \left( -\frac{1}{iQr} \right) \left[ e^{iQr} - e^{-iQr} \right]
\]

\[
= \frac{1}{2} \left[ e^{iQr} - e^{-iQr} \right]
\]

\[
= \sin(Qr)
\]

considering \( Qr \approx 0 \)
Orientation averaging

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Randomly oriented electrons

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Recall that when we had a fixed orientation of the two electrons, we had an intensity variation

\[ I(\vec{Q}) = 2r_0^2 (1 + \cos(Qr)). \]
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When we now replace the two arbitrary scattering distributions with electrons \((f_1, f_2 \to -r_0)\), we change the intensity profile significantly.
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