Today’s Outline - January 23, 2018

- Fundamental wavelength from an undulator
- Higher harmonics
- On and off-axis spectrum
- Undulator to wiggler comparison
- Undulator harmonics
- Undulator coherence
- Emittance
- Time structure
- ERLs and FELs

Reading Assignment: Chapter 3.1–3.3

Homework Assignment #01:
Chapter 2: 2,3,5,6,8
due Thursday, January 25, 2018
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For an undulator of period $\lambda_u$ we have derived the following undulator parameters and their relationships.
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the $K$ parameter, a dimensionless quantity which represents the “strength” of the undulator.
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$$= 0.934 \lambda_u [\text{cm}] B_0 [\text{T}]$$
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- The electron path length through the undulator, $S\lambda_u$
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- The electron path length through the undulator, $S\lambda_u$:
  \[ S\lambda_u \approx \lambda_u \left( 1 + \frac{1}{4} \frac{K^2}{\gamma^2} \right) \]
For an undulator of period $\lambda_u$ we have derived the following undulator parameters and their relationships:

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the electron path length through the undulator, $S\lambda_u$:

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Now let’s calculate the wavelength of the radiation generated by the undulator in the laboratory frame.
Consider an electron traveling through the undulator and emitting radiation at the first maximum excursion from the center.
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Undulator wavelength

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Consider an electron traveling through the undulator and emitting radiation at the first maximum excursion from the center.

The emitted wave travels slightly faster than the electron. It moves $cT'$ in the time the electron travels a distance $\lambda_u$ along the undulator.

The observer sees radiation with a compressed wavelength, $\lambda_1$. 

$\lambda_1$
Undulator wavelength

Consider an electron traveling through the undulator and emitting radiation at the first maximum excursion from the center.

The emitted wave travels slightly faster than the electron. It moves \( cT' \) in the time the electron travels a distance \( \lambda_u \) along the undulator.

The observer sees radiation with a compressed wavelength,

\[
\lambda_1 = cT' - \lambda_u
\]
Undulator wavelength

Consider an electron traveling through the undulator and emitting radiation at the first maximum excursion from the center.

The emitted wave travels slightly faster than the electron. It moves $cT'$ in the time the electron travels a distance $\lambda_u$ along the undulator.

The observer sees radiation with a compressed wavelength, along with harmonics which satisfy the same condition.

$$n\lambda_n = cT' - \lambda_u$$
The fundamental wavelength

The fundamental wavelength must be corrected for the observer angle $\theta$

$$\lambda_1 = cT' - \lambda_u \cos \theta$$
The fundamental wavelength

The fundamental wavelength must be corrected for the observer angle $\theta$

$$\lambda_1 = cT' - \lambda_u \cos \theta$$

Over the time $T'$ the electron actually travels a distance $S\lambda_u$, so that

$$T' = \frac{S\lambda_u}{v}$$
The fundamental wavelength

The fundamental wavelength must be corrected for the observer angle $\theta$

$$\lambda_1 = c T' - \lambda_u \cos \theta$$

$$= \lambda_u \left( S \frac{c}{v} - \cos \theta \right)$$

Over the time $T'$ the electron actually travels a distance $S\lambda_u$, so that

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Over the time $T'$ the electron actually travels a distance $S\lambda_u$, so that

$$T' = \frac{S\lambda_u}{v}$$

$$S \approx 1 + \frac{K^2}{4\gamma^2}$$
The fundamental wavelength

The fundamental wavelength must be corrected for the observer angle $\theta$

$$\lambda_1 = c T' - \lambda_u \cos \theta$$

$$= \lambda_u \left( S \frac{c}{v} - \cos \theta \right)$$

$$= \lambda_u \left( \left[ 1 + \frac{K^2}{4\gamma^2} \right] \frac{1}{\beta} - \cos \theta \right)$$

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Since $\gamma$ is large, the maximum observation angle $\theta$ is small so
The fundamental wavelength

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Since $\gamma$ is large, the maximum observation angle $\theta$ is small so

$$\lambda_1 \approx \lambda_u \left( \frac{1}{\beta} + \frac{K^2}{4\gamma^2\beta} - 1 + \frac{\theta^2}{2} \right)$$
The fundamental wavelength

The fundamental wavelength must be corrected for the observer angle \( \theta \)

\[
\lambda_1 = cT' - \lambda_u \cos \theta
\]

\[
= \lambda_u \left( S \frac{c}{v} - \cos \theta \right)
\]

\[
= \lambda_u \left( \left[ 1 + \frac{K^2}{4\gamma^2} \right] \frac{1}{\beta} - \cos \theta \right)
\]

Since \( \gamma \) is large, the maximum observation angle \( \theta \) is small so

\[
\lambda_1 \approx \lambda_u \left( \frac{1}{\beta} + \frac{K^2}{4\gamma^2\beta} - 1 + \frac{\theta^2}{2} \right) = \frac{\lambda_u}{2\gamma^2} \left( \frac{2\gamma^2}{\beta} + \frac{K^2}{2\beta} - 2\gamma^2 + \gamma^2 \theta^2 \right)
\]

Over the time \( T' \) the electron actually travels a distance \( S \lambda_u \), so that

\[
T' = \frac{S \lambda_u}{v}
\]

\[
S \approx 1 + \frac{K^2}{4\gamma^2}
\]
The fundamental wavelength

\[ \lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2\gamma^2}{\beta} + \frac{K^2}{2\beta} - 2\gamma^2 + \gamma^2\theta^2 \right) \]
The fundamental wavelength

\[ \lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2\gamma^2}{\beta} + \frac{K^2}{2\beta} - 2\gamma^2 + \gamma^2 \theta^2 \right) \]

regrouping terms
The fundamental wavelength

\[ \lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2\gamma^2}{\beta} + \frac{K^2}{2\beta} - 2\gamma^2 + \gamma^2 \theta^2 \right) \]

regrouping terms

\[ \approx \frac{\lambda_u}{2\gamma^2} \left( 2\gamma^2 \left[ \frac{1}{\beta} - 1 \right] + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right) \]
The fundamental wavelength

\[ \lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2\gamma^2}{\beta} + \frac{K^2}{2\beta} - 2\gamma^2 + \gamma^2\theta^2 \right) \]

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\[ \gamma = \sqrt{\frac{1}{1 - \beta^2}} \]
The fundamental wavelength

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\[ \gamma = \sqrt{\frac{1}{1 - \beta^2}} \]

\[ 1 - \beta^2 = (1 + \beta)(1 - \beta) \]
The fundamental wavelength

\[ \lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2\gamma^2}{\beta} + \frac{K^2}{2\beta} - 2\gamma^2 + \gamma^2\theta^2 \right) \]

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\[ \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta(1 + \beta)} + \frac{K^2}{2\beta} - [\gamma\theta]^2 \right) \]

regrouping terms

\[ \gamma = \sqrt{\frac{1}{1 - \beta^2}} \]

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The fundamental wavelength

\[ \lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta(1 + \beta)} + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right) \]
The fundamental wavelength

If we assume that $\beta \sim 1$ for these highly relativistic electrons

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta(1+\beta)} + \frac{K^2}{2\beta} - (\gamma\theta)^2 \right)$$
The fundamental wavelength

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and directly on axis

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right)$$
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$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right)$$

for a typical undulator $\gamma \sim 10^4$, $K \sim 1$, and $\lambda_u \sim 2\text{cm}$ so we estimate

$$\lambda_1 \approx \frac{2 \times 10^{-2}}{2 (10^4)^2} \left( 1 + \frac{(1)^2}{2} \right)$$
The fundamental wavelength

If we assume that $\beta \sim 1$ for these highly relativistic electrons

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta(1+\beta)} + \frac{K^2}{2\beta} - (\gamma\theta)^2 \right) \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2\beta} - (\gamma\theta)^2 \right)$$

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for a typical undulator $\gamma \sim 10^4$, $K \sim 1$, and $\lambda_u \sim 2\text{cm}$ so we estimate

$$\lambda_1 \approx \frac{2 \times 10^{-2}}{2 \left(10^4\right)^2} \left( 1 + \frac{(1)^2}{2} \right) = 1.5 \times 10^{-10}\text{m} = 1.5\text{Å}$$
The fundamental wavelength

If we assume that $\beta \sim 1$ for these highly relativistic electrons

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta(1+\beta)} + \frac{K^2}{2\beta} - (\gamma\theta)^2 \right) \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2\beta} - (\gamma\theta)^2 \right)$$

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This corresponds to an energy $\mathcal{E}_1 \approx 8.2\text{keV}$ but as the undulator gap is widened
The fundamental wavelength

If we assume that $\beta \sim 1$ for these highly relativistic electrons

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta(1 + \beta)} + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right) \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right)$$

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This corresponds to an energy $\mathcal{E}_1 \approx 8.2\text{keV}$ but as the undulator gap is widened, $B_0$ decreases
The fundamental wavelength

If we assume that $\beta \sim 1$ for these highly relativistic electrons

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta(1 + \beta)} + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right) \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right)$$

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$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right)$$

for a typical undulator $\gamma \sim 10^4$, $K \sim 1$, and $\lambda_u \sim 2$cm so we estimate

$$\lambda_1 \approx \frac{2 \times 10^{-2}}{2 (10^4)^2} \left( 1 + \frac{(1)^2}{2} \right) = 1.5 \times 10^{-10} \text{m} = 1.5\text{Å}$$

This corresponds to an energy $E_1 \approx 8.2$keV but as the undulator gap is widened, $B_0$ decreases, $K$ decreases.
The fundamental wavelength

If we assume that $\beta \sim 1$ for these highly relativistic electrons

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( \frac{2}{\beta (1 + \beta)} + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right) \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2\beta} - (\gamma \theta)^2 \right)$$

and directly on axis

$$\lambda_1 \approx \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right)$$

for a typical undulator $\gamma \sim 10^4$, $K \sim 1$, and $\lambda_u \sim 2\text{cm}$ so we estimate

$$\lambda_1 \approx \frac{2 \times 10^{-2}}{2 (10^4)^2} \left( 1 + \frac{(1)^2}{2} \right) = 1.5 \times 10^{-10} \text{m} = 1.5\text{Å}$$

This corresponds to an energy $\mathcal{E}_1 \approx 8.2\text{keV}$ but as the undulator gap is widened, $B_0$ decreases, $K$ decreases, $\lambda_1$ decreases.
The fundamental wavelength

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This corresponds to an energy $\mathcal{E}_1 \approx 8.2\text{keV}$ but as the undulator gap is widened, $B_0$ decreases, $K$ decreases, $\lambda_1$ decreases, and $\mathcal{E}_1$ increases.
Recall that we developed an expression for the Doppler time compression of the emission from a moving electron as a function of the observer angle.
Recall that we developed an expression for the Doppler time compression of the emission from a moving electron as a function of the observer angle.

\[
\frac{dt}{dt'} = 1 - \vec{n} \cdot \vec{\beta}(t')
\]
Recall that we developed an expression for the Doppler time compression of the emission from a moving electron as a function of the observer angle. This can be rewritten in terms of the coordinates in the figure using the vector of unit length in the observer direction:

\[
\frac{dt}{dt'} = 1 - \vec{n} \cdot \vec{\beta}(t')
\]

\[
\vec{n} = \left\{ \phi, \psi, \sqrt{1 - \theta^2} \right\}
\]

\[
\vec{\beta} = \beta \left\{ \alpha, 0, \sqrt{1 - \alpha^2} \right\}
\]
Recall that we developed an expression for the Doppler time compression of the emission from a moving electron as a function of the observer angle. This can be rewritten in terms of the coordinates in the figure using the vector of unit length in the observer direction:

\[
\frac{dt}{dt'} = 1 - \vec{n} \cdot \vec{\beta}(t')
\]

\[
\vec{n} \approx \{ \phi, \psi, (1 - \theta^2/2) \}
\]

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Recall that we developed an expression for the Doppler time compression of the emission from a moving electron as a function of the observer angle.
This can be rewritten in terms of the coordinates in the figure using the vector of unit length in the observer direction:

\[
\frac{dt}{dt'} = 1 - \vec{n} \cdot \vec{\beta}(t')
\]

\[
\approx 1 - \beta \left[ \alpha \phi + \left( 1 - \frac{\theta^2}{2} - \frac{\alpha^2}{2} \right) \right]
\]

\[
\vec{n} \approx \left\{ \phi, \psi, (1 - \theta^2/2) \right\}
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\vec{n} \approx \{ \phi, \psi, (1 - \theta^2/2) \}
\]

\[
\vec{\beta} \approx \beta \{ \alpha, 0, (1 - \alpha^2/2) \}
\]

\[
\frac{dt}{dt'} \approx 1 - \left( 1 - \frac{1}{2\gamma^2} \right) \left( 1 + \alpha \phi - \frac{\theta^2}{2} - \frac{\alpha^2}{2} \right)
\]
Higher harmonics

\[
\frac{dt}{dt'} \approx 1 - \left(1 - \frac{1}{2\gamma^2}\right) \left(1 + \alpha \phi - \frac{\theta^2}{2} - \frac{\alpha^2}{2}\right)
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This differential equation can be solved, realizing that \( \phi \) and \( \theta \) are constant while \( \alpha(t') \) varies as the electron moves through the insertion device, and gives:
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\[\omega_1 \gg \omega_u\] as expected because of the Doppler compression
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This differential equation can be solved, realizing that \( \phi \) and \( \theta \) are constant while \( \alpha(t') \) varies as the electron moves through the insertion device, and gives:

\[ \omega_1 t = \omega_u t' - \frac{K^2/4}{1 + (\gamma \theta)^2 + K^2/2} \sin \left( 2\omega_u t' \right) \]

\[ \omega_1 \gg \omega_u \text{ as expected because of the Doppler compression, but they are not proportional because of the second} \]
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\(\omega_1 \gg \omega_u\) as expected because of the Doppler compression, but they are not proportional because of the second and third terms.
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The motion of the electron, \(\sin \omega_u t'\), is always sinusoidal, but because of the additional terms, the motion as seen by the observer, \(\sin \omega_1 t\), is not.
On-axis undulator characteristics

\[ \omega_1 t = \omega_u t' - \frac{K^2/4}{1 + (\gamma \theta)^2 + K^2/2} \sin(2\omega_u t') \]

Suppose we have \( K = 1 \) and \( \theta = 0 \) (on axis), then
On-axis undulator characteristics

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Suppose we have \( K = 1 \) and \( \theta = 0 \) (on axis), then

\[ \omega_1 t = \omega_u t' + \frac{1}{6} \sin (2\omega_u t') \]
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Similarly, for \( K = 2 \) and \( K = 5 \), the deviation becomes more pronounced.
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Similarly, for \( K = 2 \) and \( K = 5 \), the deviation becomes more pronounced. This shows how higher harmonics must be present in the radiation as seen by the observer.
Off-axis undulator characteristics

\[ \omega_1 t = \omega_u t' - \frac{K^2/4}{1 + (\gamma \theta)^2 + K^2/2} \sin(2\omega_u t') - \frac{2K\gamma}{1 + (\gamma \theta)^2 + K^2/2} \phi \sin(\omega_u t') \]

When \( K = 2 \) and \( \theta = \phi = 1/\gamma \), we have
Off-axis undulator characteristics

$$\omega_1 t = \omega_u t' - \frac{K^2/4}{1 + (\gamma \theta)^2 + K^2/2} \sin(2\omega_u t') - \frac{2K\gamma}{1 + (\gamma \theta)^2 + K^2/2} \phi \sin (\omega_u t')$$

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The last term introduces an antisymmetric term which skews the function.
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The last term introduces an antisymmetric term which skews the function and leads to the presence of forbidden harmonics (2\(^{nd}\), 4\(^{th}\), etc) in the radiation from the undulator compared to the on-axis radiation.
Spectral comparison

- Brilliance is 6 orders larger than a bending magnet.
- Both odd and even harmonics appear.
- Harmonics can be tuned in energy (dashed lines).

![Graph showing Brilliance vs Photon Energy for Undulator, Wiggler, and Bending magnet.](Image)
Spectral comparison

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Diffraction grating

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$$\sum_{m=0}^{N-1} e^{i(\vec{k} \cdot \vec{r} + 2\pi m \epsilon)}$$
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Geometric series

The sum is simply a geometric series, \( S_N \) with \( k = e^{i2\pi \epsilon} \).
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$$S_N = \sum_{m=0}^{N-1} k^m = 1 + k + k^2 + \cdots + k^{N-2} + k^{N-1}$$
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$$S_N - kS_N = 1 - k^N \quad \rightarrow \quad S_N = \frac{1 - k^N}{1 - k}$$
Intensity from a diffraction grating

Restoring the expression for \( k = e^{i2\pi \epsilon} \), we have:
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Intensity from a diffraction grating

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I = \frac{\sin^2 (\pi N \epsilon)}{\sin^2 (\pi \epsilon)}
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Beam coherence

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Beam coherence

An $N$ period undulator is basically like a diffraction grating, only in the time domain rather than the space domain.

$2\pi \varepsilon = 0$

![Graph showing intensity vs. $2\pi \varepsilon$]
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$$2\pi \epsilon = 5^\circ$$
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![Diagram of undulator and diffraction pattern](image)

\[ 2\pi\varepsilon = 20^\circ \]

- **Graph:**
  - X-axis: $2\pi\varepsilon$ [degrees]
  - Y-axis: Intensity (arb units)

- With the height and width of the peak dependent on the number of poles.
Beam coherence

An $N$ period undulator is basically like a diffraction grating, only in the time domain rather than the space domain.

\[ 2\pi \epsilon = 25^\circ \]

With the height and width of the peak dependent on the number of poles.
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\[ 2\pi \varepsilon = 30^\circ \]
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\[ 2\pi \varepsilon = 45^\circ \]

![Graph showing intensity versus $2\pi \varepsilon$ degrees]

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The more poles in the undulator, the more monochromatic the beam.
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Higher order harmonics have narrower energy bandwidth but lower peak intensity as seen from the phasor diagram representation.
There are two important time scales for a storage ring such as the APS: pulse length and interpulse spacing.

Synchrotron time structure

![Graph showing intensity vs. time (μsec) with pulse lengths and interpulse spacings indicated.](image)
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The APS pulse length in 24-bunch mode is 90 ps while the pulses come every 154 ns.
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The APS pulse length in 24-bunch mode is 90 ps while the pulses come every 154 ns.

Other modes include single-bunch mode for timing experiments and 324-bunch mode (inter pulse timing of 11.7 ns) for a more constant x-ray flux.
Emittance

Is there a limit to the brilliance of an undulator source at a synchrotron?

The brilliance is inversely proportional to the square of the product of the linear source size and the angular divergence:

$$\text{brilliance} = \frac{\text{flux}}{\text{photons/s}} \cdot \frac{\text{divergence}}{\text{mrad}^2} \cdot \text{source size [mm}^2\text{]} \cdot 0.1\% \text{ bandwidth}$$

The product of the source size and divergence is called the emittance, $\epsilon$, and the brilliance is thus limited by the product of the emittance of the radiation in the horizontal and vertical directions $\epsilon_x \epsilon_y$.

This emittance cannot be changed but it can be rotated or deformed by magnetic fields as the electron beam travels around the storage ring as long as the area is kept constant.
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For photon emission from a single electron in a 2m undulator at 1Å
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Evolution of APS parameters

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When first commissioned in 1995, the APS electron beam size and divergence was relatively large, particularly in the horizontal, $x$ direction. By the end of the first decade of operation, the horizontal source size decreased by about 16% and its horizontal divergence by more than 50%. At the same time, the vertical source size decreased by over 90% and the vertical divergence by nearly 67%. The next big upgrade (slated for 2022) will make the beam more square in space and by choosing the undulator correctly, a higher performance insertion device.
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