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The recursive relation can be seen from the calculation of reflectivity of the next layer up

$$r_{N-2,N-1} = \frac{r'_{N-2,N-1} + r_{N-1,N} p_{N-1}^2}{1 + r'_{N-2,N-1} r_{N-1,N} p_{N-1}^2}$$

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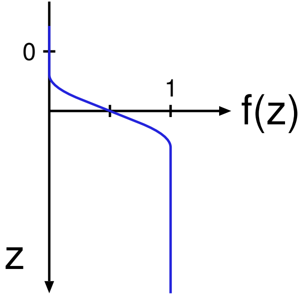

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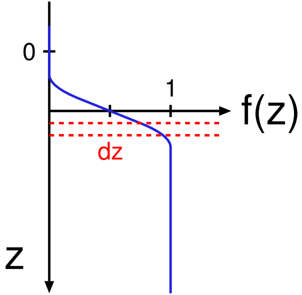

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The reflectivity can be computed as the superposition of the reflectivity of an infinitesmal slab of thickness dz at a depth z.

The differential reflectivity from a slab of thickness dz at depth z is.

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Calculating the full reflection coefficient relative to the Fresnel reflection coefficient

$$\frac{R(Q)}{R_F(Q)} = \left|\int_0^\infty \left(\frac{df}{dz}\right) e^{iQz}dz\right|^2$$

The Error Function - a Specific Case

The error function is often chosen as a model for the density gradient

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whose Fourier transform is also a Gaussian, which when squared to obtain the reflection coefficient, gives

$$R(Q) = R_F(Q) e^{-Q^2\sigma^2}$$

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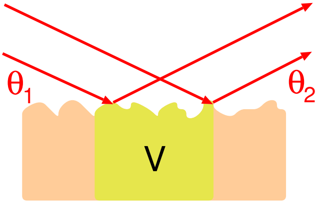

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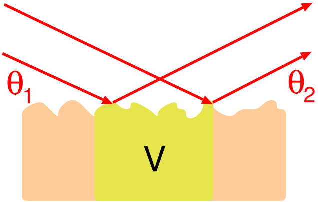

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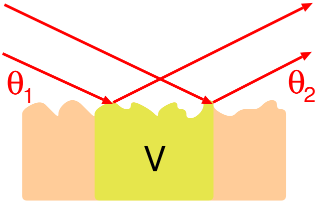

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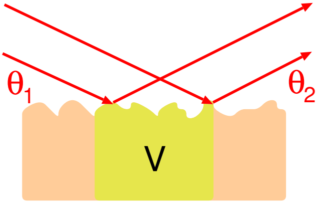

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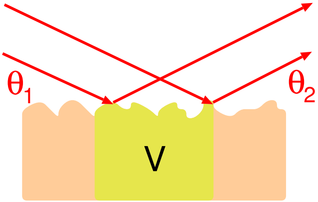

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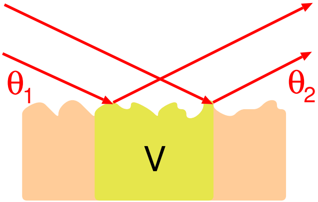

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The actual scattering cross section is the square of this integral

$$\frac{d\sigma}{d\Omega} = \left(\frac{r_o \rho}{Q_z} \right)^2 \int_V \int_{V'} e^{iQ_z(h(x,y) - h(x',y'))} e^{iQ_x(x-x')} e^{iQ_y(y-y')} dxdydx'dy'$$

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