# Today's outline - April 14, 2022

# Today's outline - April 14, 2022

- Quantum error correcting codes

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks
- Examples

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks
- Examples

Reading assignment: 11.3

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks
- Examples

Reading assignment: 11.3

Homework Assignment #07:
Chapter 9:2,3,4; Chapter 10:3,4,11
Due Thursday, April 21, 2022

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks
- Examples

Reading assignment: 11.3

Homework Assignment #07:
Chapter 9:2,3,4; Chapter 10:3,4,11
Due Thursday, April 21, 2022

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks
- Examples

Reading assignment: 11.3

Exam #2 - April 26, 2022

Homework Assignment #07:
Chapter 9:2,3,4; Chapter 10:3,4,11
Due Thursday, April 21, 2022

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks
- Examples

Reading assignment: 11.3

Homework Assignment #07:
Chapter 9:2,3,4; Chapter 10:3,4,11
Due Thursday, April 21, 2022

Exam #2 - April 26, 2022

Final Exam will be 10 minute presentations on a quantum computing journal article

# Today's outline - April 14, 2022

- Quantum error correcting codes
- Correctable sets of errors
- Diagnosing errors
- Error correction across blocks
- Examples

Reading assignment: 11.3

Homework Assignment #07:
Chapter 9:2,3,4; Chapter 10:3,4,11
Due Thursday, April 21, 2022

Exam #2 - April 26, 2022

Final Exam will be 10 minute presentations on a quantum computing journal article

Get my approval by Friday, April 15, 2022

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n, k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n, k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n - k$ qubits are $|0\rangle$

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n,k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n-k$ qubits are $|0\rangle$

Any unitary transformation $U_C : W \to C$ is a possible encoding operator for code $C$

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n, k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n - k$ qubits are $|0\rangle$

Any unitary transformation $U_C : W \rightarrow C$ is a possible encoding operator for code $C$

Elements $|w\rangle \in W$ are called message words and elements of $C$ are called codewords

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n, k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n - k$ qubits are $|0\rangle$

Any unitary transformation $U_C : W \to C$ is a possible encoding operator for code $C$

Elements $|w\rangle \in W$ are called message words and elements of $C$ are called codewords

In this terminology the $[[3, 1]]$ code is spanned by $\{|000\rangle, |111\rangle\}$ and the logical qubits are obtained by

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n, k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n-k$ qubits are $|0\rangle$

Any unitary transformation $U_C : W \to C$ is a possible encoding operator for code $C$

Elements $|w\rangle \in W$ are called message words and elements of $C$ are called codewords

In this terminology the $[[3, 1]]$ code is spanned by $\{|000\rangle, |111\rangle\}$ and the logical qubits are obtained by

$$U_C : |000\rangle = |0\rangle \mapsto |000\rangle = |\tilde{0}\rangle$$

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n, k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n-k$ qubits are $|0\rangle$

Any unitary transformation $U_C : W \to C$ is a possible encoding operator for code $C$

Elements $|w\rangle \in W$ are called message words and elements of $C$ are called codewords

In this terminology the $[[3, 1]]$ code is spanned by $\{|000\rangle, |111\rangle\}$ and the logical qubits are obtained by

$$U_C : |000\rangle = |0\rangle \mapsto |000\rangle = |\tilde{0}\rangle$$
$$|001\rangle = |1\rangle \mapsto |111\rangle = |\tilde{1}\rangle$$

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n,k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n-k$ qubits are $|0\rangle$

Any unitary transformation $U_C : W \to C$ is a possible encoding operator for code $C$

Elements $|w\rangle \in W$ are called message words and elements of $C$ are called codewords

In this terminology the $[[3,1]]$ code is spanned by $\{|000\rangle, |111\rangle\}$ and the logical qubits are obtained by

$$U_C : |000\rangle = |0\rangle \mapsto |000\rangle = |\tilde{0}\rangle$$
$$|001\rangle = |1\rangle \mapsto |111\rangle = |\tilde{1}\rangle$$

Note that we have only defined the action of $U_C$ on the states in $W$ where only the low order qubit can be non-zero

# Quantum error correcting codes

The formalism for quantum error correction codes is analogous to that for classical systems

A $[[n, k]]$ quantum block code $C$ is a $2^k$-dimensional subspace of the $n$-qubit space $V$ with $2^n$ dimensions

Subspace $W$ is a $k$-qubit message space where the first $n-k$ qubits are $|0\rangle$

Any unitary transformation $U_C : W \to C$ is a possible encoding operator for code $C$

Elements $|w\rangle \in W$ are called message words and elements of $C$ are called codewords

In this terminology the $[[3, 1]]$ code is spanned by $\{|000\rangle, |111\rangle\}$ and the logical qubits are obtained by

$$U_C : |000\rangle = |0\rangle \mapsto |000\rangle = |\tilde{0}\rangle$$
$$|001\rangle = |1\rangle \mapsto |111\rangle = |\tilde{1}\rangle$$

Note that we have only defined the action of $U_C$ on the states in $W$ where only the low order qubit can be non-zero

$U_C$ may well have some action on other states not in $W$ but this is generally ignored

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \ldots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \ldots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \ldots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \ldots, E_L\},$$

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \ldots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \ldots, E_L\}, \quad E_i : V \to V$$

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a | E_i^\dagger E_j | c_b \rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a | E_i^\dagger E_j | c_b \rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

When $i = j$ this leads to the orthogonality condition

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a|E_i^\dagger E_j|c_b\rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

When $i = j$ this leads to the orthogonality condition

$$\langle c_a|c_b\rangle = 0$$

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a|E_i^\dagger E_j|c_b\rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

When $i = j$ this leads to the orthogonality condition

$$\langle c_a|c_b\rangle = 0$$

Furthermore, any measurement made to determine the error must not give any information about the logical state

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a | E_i^\dagger E_j | c_b \rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

When $i = j$ this leads to the orthogonality condition

$$\langle c_a | c_b \rangle = 0$$

Furthermore, any measurement made to determine the error must not give any information about the logical state

$$\langle c_a | E_i^\dagger E_j | c_a \rangle = \langle c_b | E_i^\dagger E_j | c_b \rangle; \quad c_a, c_b \in C$$

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a|E_i^\dagger E_j|c_b\rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

When $i = j$ this leads to the orthogonality condition

$$\langle c_a|c_b\rangle = 0$$

Furthermore, any measurement made to determine the error must not give any information about the logical state

$$\langle c_a|E_i^\dagger E_j|c_a\rangle = \langle c_b|E_i^\dagger E_j|c_b\rangle; \quad c_a, c_b \in C$$

$\mathcal{E}$ is a correctable set of errors if

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \ldots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \ldots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a|E_i^\dagger E_j|c_b\rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

When $i = j$ this leads to the orthogonality condition

$$\langle c_a|c_b\rangle = 0$$

Furthermore, any measurement made to determine the error must not give any information about the logical state

$$\langle c_a|E_i^\dagger E_j|c_a\rangle = \langle c_b|E_i^\dagger E_j|c_b\rangle; \quad c_a, c_b \in C$$

$\mathcal{E}$ is a correctable set of errors if

$$\langle c_a|E_i^\dagger E_j|c_b\rangle = m_{ij}\delta_{ab}$$

# Correctable sets of errors

Let $B_C = \{|c_0\rangle, \dots, |c_{k-1}\rangle\}$ be an orthonormal basis for $C$

If $\mathcal{E}$ is a finite set of unitary error transformations

$$\mathcal{E} = \{E_1, E_2, \dots, E_L\}, \quad E_i : V \to V$$

In order to be able to correct errors, two distinct error transformations in $\mathcal{E}$ must take different codewords to orthogonal states that can be distinguished by measurement

This can be written

$$\langle c_a|E_i^\dagger E_j|c_b\rangle = 0, \quad E_i, E_j \in \mathcal{E}; \quad c_a, c_b \in C$$

When $i = j$ this leads to the orthogonality condition

$$\langle c_a|c_b\rangle = 0$$

Furthermore, any measurement made to determine the error must not give any information about the logical state

$$\langle c_a|E_i^\dagger E_j|c_a\rangle = \langle c_b|E_i^\dagger E_j|c_b\rangle; \quad c_a, c_b \in C$$

$\mathcal{E}$ is a correctable set of errors if

$$\langle c_a|E_i^\dagger E_j|c_b\rangle = m_{ij}\delta_{ab}$$

Codes not satisfying this condition are said to be degenerate

# Example 11.2.6

For the [[3,1]] code and a set of errors $\mathcal{E} = \{E_{ij}\}$ the correctable error set for a single-qubit bit-flip error is

# Example 11.2.6

For the [[3,1]] code and a set of errors $\mathcal{E} = \{E_{ij}\}$ the correctable error set for a single-qubit bit-flip error is

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

# Example 11.2.6

For the [[3,1]] code and a set of errors $\mathcal{E} = \{E_{ij}\}$ the correctable error set for a single-qubit bit-flip error is

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

One of the many other correctable error sets could be $\mathcal{E}' = \{E'_{ij}\}$ where

# Example 11.2.6

For the [[3,1]] code and a set of errors $\mathcal{E} = \{E_{ij}\}$ the correctable error set for a single-qubit bit-flip error is

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

One of the many other correctable error sets could be $\mathcal{E}' = \{E'_{ij}\}$ where

$$E'_{00} = I \otimes I \otimes I, \quad E'_{01} = X \otimes X \otimes I, \quad E'_{10} = X \otimes I \otimes X, \quad E'_{11} = I \otimes X \otimes X$$

# Example 11.2.6

For the [[3,1]] code and a set of errors $\mathcal{E} = \{E_{ij}\}$ the correctable error set for a single-qubit bit-flip error is

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

One of the many other correctable error sets could be $\mathcal{E}' = \{E'_{ij}\}$ where

$$E'_{00} = I \otimes I \otimes I, \quad E'_{01} = X \otimes X \otimes I, \quad E'_{10} = X \otimes I \otimes X, \quad E'_{11} = I \otimes X \otimes X$$

The latter is the correctable error set for all two-qubit bit flips

# Example 11.2.6

For the [[3,1]] code and a set of errors $\mathcal{E} = \{E_{ij}\}$ the correctable error set for a single-qubit bit-flip error is

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

One of the many other correctable error sets could be $\mathcal{E}' = \{E'_{ij}\}$ where

$$E'_{00} = I \otimes I \otimes I, \quad E'_{01} = X \otimes X \otimes I, \quad E'_{10} = X \otimes I \otimes X, \quad E'_{11} = I \otimes X \otimes X$$

The latter is the correctable error set for all two-qubit bit flips

One must choose one of the two sets to correct for and generally single-qubit bit-flips are much more probable than two-qubit bit-flips

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n, k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \leq i < M$

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n, k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \leq i < M$

Applying the error transformation $E_s$ to a state $|v\rangle$

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n, k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \leq i < M$

Applying the error transformation $E_s$ to a state $|v\rangle$

$$E_s|v\rangle \mapsto |w\rangle, \quad |v\rangle \in C, \quad E_s \in \mathcal{E}$$

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n,k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \leq i < M$

Applying the error transformation $E_s$ to a state $|v\rangle$

$$E_s|v\rangle \mapsto |w\rangle, \quad |v\rangle \in C, \quad E_s \in \mathcal{E}$$

The subspaces $E_iC$ and $E_jC$ are orthogonal for all $i \neq j$, so there is a unique mapping of $E_s|v\rangle$ to $|w\rangle$

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n, k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \leq i < M$

Applying the error transformation $E_s$ to a state $|v\rangle$

$$E_s|v\rangle \mapsto |w\rangle, \quad |v\rangle \in C, \quad E_s \in \mathcal{E}$$

The subspaces $E_i C$ and $E_j C$ are orthogonal for all $i \neq j$, so there is a unique mapping of $E_s|v\rangle$ to $|w\rangle$

Once the subspace $E_s C$ which contains $|w\rangle$ is identified, the error may be corrected by computing $E_s^\dagger|w\rangle \mapsto |v\rangle$

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n, k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \leq i < M$

Applying the error transformation $E_s$ to a state $|v\rangle$

$$E_s|v\rangle \mapsto |w\rangle, \quad |v\rangle \in C, \quad E_s \in \mathcal{E}$$

The subspaces $E_i C$ and $E_j C$ are orthogonal for all $i \neq j$, so there is a unique mapping of $E_s|v\rangle$ to $|w\rangle$

Once the subspace $E_s C$ which contains $|w\rangle$ is identified, the error may be corrected by computing $E_s^\dagger|w\rangle \mapsto |v\rangle$

Thus the goal is to measure the error subspace which contains $|w\rangle$ without measuring $|w\rangle$ itself

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n, k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \le i < M$

Applying the error transformation $E_s$ to a state $|v\rangle$

$$E_s|v\rangle \mapsto |w\rangle, \quad |v\rangle \in C, \quad E_s \in \mathcal{E}$$

The subspaces $E_i C$ and $E_j C$ are orthogonal for all $i \neq j$, so there is a unique mapping of $E_s|v\rangle$ to $|w\rangle$

Once the subspace $E_s C$ which contains $|w\rangle$ is identified, the error may be corrected by computing $E_s^\dagger|w\rangle \mapsto |v\rangle$

Thus the goal is to measure the error subspace which contains $|w\rangle$ without measuring $|w\rangle$ itself

$$W = \bigoplus_{i=0}^{M-1} W_i, \quad W_i = E_i C$$

# Diagnosing and correcting errors

Suppose that $C$ is a $[[n,k]]$ quantum code that is nondegenerate with respect to a correctable error set $\mathcal{E} = \{E_i\}, 0 \le i < M$

Applying the error transformation $E_s$ to a state $|v\rangle$

$$E_s|v\rangle \mapsto |w\rangle, \quad |v\rangle \in C, \quad E_s \in \mathcal{E}$$

The subspaces $E_i C$ and $E_j C$ are orthogonal for all $i \neq j$, so there is a unique mapping of $E_s|v\rangle$ to $|w\rangle$

Once the subspace $E_s C$ which contains $|w\rangle$ is identified, the error may be corrected by computing $E_s^\dagger|w\rangle \mapsto |v\rangle$

Thus the goal is to measure the error subspace which contains $|w\rangle$ without measuring $|w\rangle$ itself

$$W = \bigoplus_{i=0}^{M-1} W_i, \quad W_i = E_i C$$

Let $W_M$ be the subspace of $V$ which is orthogonal to $W$, sincd the $W_i$ are mutually orthogonal there is an observable $O$ with eigensubspaces which are the $W_i$

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle$$

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

Thus applying the operator $E_s^\dagger$ will recover the corrected state:

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

Thus applying the operator $E_s^\dagger$ will recover the corrected state: $E_s^\dagger|w\rangle = E_s^\dagger E_s|v\rangle = |v\rangle$

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

Thus applying the operator $E_s^\dagger$ will recover the corrected state: $E_s^\dagger|w\rangle = E_s^\dagger E_s|v\rangle = |v\rangle$

$U_P$ is a syndrome extraction operator which places the value $s$ into the $m$-qubit ancilla register

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n + m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

Thus applying the operator $E_s^\dagger$ will recover the corrected state: $E_s^\dagger|w\rangle = E_s^\dagger E_s|v\rangle = |v\rangle$

$U_P$ is a syndrome extraction operator which places the value $s$ into the $m$-qubit ancilla register

Each qubit of the $m$-qubit register corresponds to a binary observable

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n+m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

Thus applying the operator $E_s^\dagger$ will recover the corrected state: $E_s^\dagger|w\rangle = E_s^\dagger E_s|v\rangle = |v\rangle$

$U_P$ is a syndrome extraction operator which places the value $s$ into the $m$-qubit ancilla register

Each qubit of the $m$-qubit register corresponds to a binary observable

Ancilla qubits permit the conversion of non-unitary errors to unitary errors which can be corrected

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n+m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

Thus applying the operator $E_s^\dagger$ will recover the corrected state: $E_s^\dagger|w\rangle = E_s^\dagger E_s|v\rangle = |v\rangle$

$U_P$ is a syndrome extraction operator which places the value $s$ into the $m$-qubit ancilla register

Each qubit of the $m$-qubit register corresponds to a binary observable

Ancilla qubits permit the conversion of non-unitary errors to unitary errors which can be corrected

As an alternative, ancilla qubits can be used as control qubits to directly apply the correction without needing to be measured

# Dignosing and correcting errors

If $P_i$ is the projector of $O$ onto subspace $W_i$ and $m = \log_2 M$, $U_P$ is a unitary operator on $n+m$ qubits

$$U_P : |w\rangle|0\rangle \mapsto \sum_{j=0}^{M-1} P_j|w\rangle|j\rangle = \sum_{j=0}^{M-1} b_j|w_j\rangle|j\rangle$$

By measuring the $m$ ancilla qubits, the error syndrome $s$ identifying the subspace of the error state $W_s = E_s C$

Thus applying the operator $E_s^\dagger$ will recover the corrected state: $E_s^\dagger|w\rangle = E_s^\dagger E_s|v\rangle = |v\rangle$

$U_P$ is a syndrome extraction operator which places the value $s$ into the $m$-qubit ancilla register

Each qubit of the $m$-qubit register corresponds to a binary observable

Ancilla qubits permit the conversion of non-unitary errors to unitary errors which can be corrected

As an alternative, ancilla qubits can be used as control qubits to directly apply the correction without needing to be measured

$$V_P = \sum_{s=0}^{M-1} E_s^\dagger \otimes |s\rangle\langle s|$$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$$W_{00} = E_{00}C$$
$$B_{00} = \{|000\rangle, |111\rangle\}$$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$$W_{00} = E_{00}C \qquad W_{01} = E_{01}C$$

$$B_{00} = \{|000\rangle, |111\rangle\} \qquad B_{01} = \{|001\rangle, |110\rangle\}$$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$$W_{00} = E_{00}C \qquad W_{01} = E_{01}C \qquad W_{10} = E_{10}C$$

$$B_{00} = \{|000\rangle, |111\rangle\} \qquad B_{01} = \{|001\rangle, |110\rangle\} \qquad B_{10} = \{|010\rangle, |101\rangle\}$$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$$W_{00} = E_{00}C \qquad W_{01} = E_{01}C \qquad W_{10} = E_{10}C \qquad W_{11} = E_{11}C$$

$$B_{00} = \{|000\rangle, |111\rangle\} \qquad B_{01} = \{|001\rangle, |110\rangle\} \qquad B_{10} = \{|010\rangle, |101\rangle\} \qquad B_{00} = \{|100\rangle, |011\rangle\}$$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$W_{00} = E_{00}C$ $\quad$ $W_{01} = E_{01}C$ $\quad$ $W_{10} = E_{10}C$ $\quad$ $W_{11} = E_{11}C$

$B_{00} = \{|000\rangle, |111\rangle\}$ $\quad$ $B_{01} = \{|001\rangle, |110\rangle\}$ $\quad$ $B_{10} = \{|010\rangle, |101\rangle\}$ $\quad$ $B_{00} = \{|100\rangle, |011\rangle\}$

The syndrome extraction operator for $C$ with error set $\mathcal{E}$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$$W_{00} = E_{00}\,C \qquad W_{01} = E_{01}\,C \qquad W_{10} = E_{10}\,C \qquad W_{11} = E_{11}\,C$$

$$B_{00} = \{|000\rangle, |111\rangle\} \qquad B_{01} = \{|001\rangle, |110\rangle\} \qquad B_{10} = \{|010\rangle, |101\rangle\} \qquad B_{00} = \{|100\rangle, |011\rangle\}$$

The syndrome extraction operator for $C$ with error set $\mathcal{E}$

$$U_P : |b_2, b_1, b_0, 0, 0\rangle \rightarrow |b_2, b_1, b_0, a_1 = b_2 \oplus b_1, a_2 = b_2 \oplus b_0\rangle$$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$$W_{00} = E_{00}C \qquad W_{01} = E_{01}C \qquad W_{10} = E_{10}C \qquad W_{11} = E_{11}C$$

$$B_{00} = \{|000\rangle, |111\rangle\} \qquad B_{01} = \{|001\rangle, |110\rangle\} \qquad B_{10} = \{|010\rangle, |101\rangle\} \qquad B_{00} = \{|100\rangle, |011\rangle\}$$

The syndrome extraction operator for $C$ with error set $\mathcal{E}$

$$U_P : |b_2, b_1, b_0, 0, 0\rangle \rightarrow |b_2, b_1, b_0, a_1 = b_2 \oplus b_1, a_2 = b_2 \oplus b_0\rangle$$

Bit $a_1$ distinguishes between eigenspaces spanned by subspaces $\{W_{00}, W_{01}\}$ and $\{W_{10}, W_{11}\}$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$W_{00} = E_{00}C$ | $W_{01} = E_{01}C$ | $W_{10} = E_{10}C$ | $W_{11} = E_{11}C$

$B_{00} = \{|000\rangle, |111\rangle\}$ | $B_{01} = \{|001\rangle, |110\rangle\}$ | $B_{10} = \{|010\rangle, |101\rangle\}$ | $B_{00} = \{|100\rangle, |011\rangle\}$

The syndrome extraction operator for $C$ with error set $\mathcal{E}$

$$U_P : |b_2, b_1, b_0, 0, 0\rangle \to |b_2, b_1, b_0, a_1 = b_2 \oplus b_1, a_2 = b_2 \oplus b_0\rangle$$

Bit $a_1$ distinguishes between eigenspaces spanned by subspaces $\{W_{00}, W_{01}\}$ and $\{W_{10}, W_{11}\}$

Bit $a_0$ distinguishes between eigenspaces spanned by subspaces $\{W_{00}, W_{10}\}$ and $\{W_{01}, W_{11}\}$

# Example 11.2.9

Consider the bit-flip code $C$ and the set of correctable errors $\mathcal{E} = \{E_{ij}\}$ with

$$E_{00} = I \otimes I \otimes I, \quad E_{01} = I \otimes I \otimes X, \quad E_{10} = I \otimes X \otimes I, \quad E_{11} = X \otimes I \otimes I$$

The orthogonal subspaces and bases corresponding to this error set are:

$$W_{00} = E_{00}C \qquad W_{01} = E_{01}C \qquad W_{10} = E_{10}C \qquad W_{11} = E_{11}C$$

$$B_{00} = \{|000\rangle, |111\rangle\} \qquad B_{01} = \{|001\rangle, |110\rangle\} \qquad B_{10} = \{|010\rangle, |101\rangle\} \qquad B_{00} = \{|100\rangle, |011\rangle\}$$

The syndrome extraction operator for $C$ with error set $\mathcal{E}$

$$U_P : |b_2, b_1, b_0, 0, 0\rangle \to |b_2, b_1, b_0, a_1 = b_2 \oplus b_1, a_2 = b_2 \oplus b_0\rangle$$

Bit $a_1$ distinguishes between eigenspaces spanned by subspaces $\{W_{00}, W_{01}\}$ and $\{W_{10}, W_{11}\}$

Bit $a_0$ distinguishes between eigenspaces spanned by subspaces $\{W_{00}, W_{10}\}$ and $\{W_{01}, W_{11}\}$

Measuring $a_1 = i$ and $a_0 = j$ projects the state into $W_{ij} = E_{ij}C$ and $E_{ij}^\dagger$ is the error corrector

# Example 11.2.10

The $C_{BF}$ single-qubit bit-flip correction circuit can be adapted to directly apply the correction after applying $U_{BF}$ by applying $V_{BF}$

# Example 11.2.10

The $C_{BF}$ single-qubit bit-flip correction circuit can be adapted to directly apply the correction after applying $U_{BF}$ by applying $V_{BF}$

# Example 11.2.10

The $C_{BF}$ single-qubit bit-flip correction circuit can be adapted to directly apply the correction after applying $U_{BF}$ by applying $V_{BF}$

$$V_{BF} = \sum_{s=0}^{3} E_s^\dagger \otimes |s\rangle\langle s|$$

# Example 11.2.10

The $C_{BF}$ single-qubit bit-flip correction circuit can be adapted to directly apply the correction after applying $U_{BF}$ by applying $V_{BF}$

$$V_{BF} = \sum_{s=0}^{3} E_s^\dagger \otimes |s\rangle\langle s| = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

# Example 11.2.10

The $C_{BF}$ single-qubit bit-flip correction circuit can be adapted to directly apply the correction after applying $U_{BF}$ by applying $V_{BF}$

$$V_{BF} = \sum_{s=0}^{3} E_s^\dagger \otimes |s\rangle\langle s| = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = I \otimes I \otimes I$$
$$E_1^\dagger = I \otimes I \otimes X$$
$$E_2^\dagger = I \otimes X \otimes I$$
$$E_3^\dagger = X \otimes I \otimes I$$

# Example 11.2.10

The $C_{BF}$ single-qubit bit-flip correction circuit can be adapted to directly apply the correction after applying $U_{BF}$ by applying $V_{BF}$

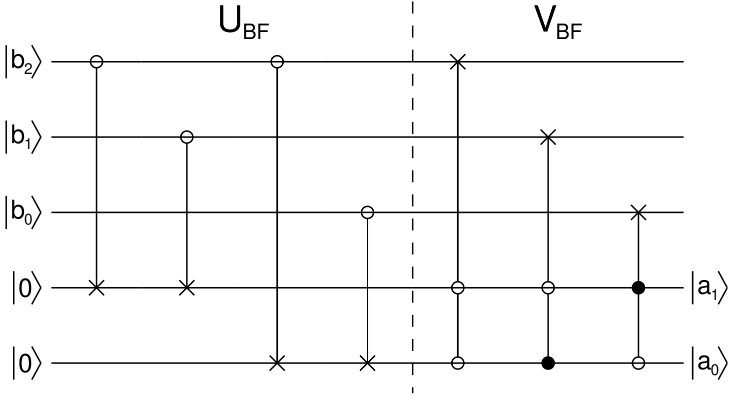
$$V_{BF} = \sum_{s=0}^{3} E_s^\dagger \otimes |s\rangle\langle s| = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = I \otimes I \otimes I$$

$$E_1^\dagger = I \otimes I \otimes X$$

$$E_2^\dagger = I \otimes X \otimes I$$

$$E_3^\dagger = X \otimes I \otimes I$$



# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\left[(E|000\rangle) \otimes |00\rangle\right] = U_{BF}\left[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\right]$$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\left[(E|000\rangle) \otimes |00\rangle\right] = U_{BF}\left[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\right] = \alpha|100\rangle|11\rangle + \beta|010\rangle|10\rangle$$

Now use $V_{BF}$ to directly correct the logical state to its initial value, noting that only one term survives for each term of $|u\rangle$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\left[(E|000\rangle) \otimes |00\rangle\right] = U_{BF}\left[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\right] = \alpha|100\rangle|11\rangle + \beta|010\rangle|10\rangle$$

Now use $V_{BF}$ to directly correct the logical state to its initial value, noting that only one term survives for each term of $|u\rangle$

$$V_{BF}|u\rangle = \left(E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|\right) \alpha|100\rangle|11\rangle +$$
$$\left(E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|\right) \beta|010\rangle|10\rangle$$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\big[(E|000\rangle) \otimes |00\rangle\big] = U_{BF}\big[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\big] = \alpha|100\rangle|11\rangle + \beta|010\rangle|10\rangle$$

Now use $V_{BF}$ to directly correct the logical state to its initial value, noting that only one term survives for each term of $|u\rangle$

$$V_{BF}|u\rangle = \big(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes \cancel{|10\rangle\langle 10|} + E_3^\dagger \otimes |11\rangle\langle 11|\big)\ \alpha|100\rangle|11\rangle +$$
$$\big(E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|\big)\ \beta|010\rangle|10\rangle$$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\left[(E|000\rangle) \otimes |00\rangle\right] = U_{BF}\left[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\right] = \alpha|100\rangle|11\rangle + \beta|010\rangle|10\rangle$$

Now use $V_{BF}$ to directly correct the logical state to its initial value, noting that only one term survives for each term of $|u\rangle$

$$\begin{aligned} V_{BF}|u\rangle = &\left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes \cancel{|10\rangle\langle 10|} + E_3^\dagger \otimes |11\rangle\langle 11|\right) \alpha|100\rangle|11\rangle + \\ &\left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes \cancel{|11\rangle\langle 11|}\right) \beta|010\rangle|10\rangle \end{aligned}$$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\left[(E|000\rangle) \otimes |00\rangle\right] = U_{BF}\left[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\right] = \alpha|100\rangle|11\rangle + \beta|010\rangle|10\rangle$$

Now use $V_{BF}$ to directly correct the logical state to its initial value, noting that only one term survives for each term of $|u\rangle$

$$\begin{aligned} V_{BF}|u\rangle = & \left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes \cancel{|10\rangle\langle 10|} + E_3^\dagger \otimes |11\rangle\langle 11|\right) \alpha|100\rangle|11\rangle + \\ & \left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes \cancel{|11\rangle\langle 11|}\right) \beta|010\rangle|10\rangle \\ = & \left(\alpha E_3^\dagger|100\rangle\right)|11\rangle + \left(\beta E_2^\dagger|010\rangle\right)|10\rangle \end{aligned}$$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\left[(E|000\rangle) \otimes |00\rangle\right] = U_{BF}\left[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\right] = \alpha|100\rangle|11\rangle + \beta|010\rangle|10\rangle$$

Now use $V_{BF}$ to directly correct the logical state to its initial value, noting that only one term survives for each term of $|u\rangle$

$$\begin{aligned} V_{BF}|u\rangle = &\left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes \cancel{|10\rangle\langle 10|} + E_3^\dagger \otimes |11\rangle\langle 11|\right) \alpha|100\rangle|11\rangle + \\ &\left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes \cancel{|11\rangle\langle 11|}\right) \beta|010\rangle|10\rangle \\ = &\left(\alpha E_3^\dagger|100\rangle\right)|11\rangle + \left(\beta E_2^\dagger|010\rangle\right)|10\rangle = |000\rangle\left(\alpha|11\rangle + \beta|10\rangle\right) \end{aligned}$$

# Example 11.2.10 (cont.)

$$V_{BF} = E_0^\dagger \otimes |00\rangle\langle 00| + E_1^\dagger \otimes |01\rangle\langle 01| + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes |11\rangle\langle 11|$$

$$E_0^\dagger = E_0 = I \otimes I \otimes I, \quad E_1^\dagger = E_1 = I \otimes I \otimes X, \quad E_2^\dagger = E_2 = I \otimes X \otimes I, \quad E_3^\dagger = E_3 = X \otimes I \otimes I$$

With the above circuit, it is possible to correct linear combinations of errors

Suppose we start with a state $|\tilde{0}\rangle = |000\rangle$ and there is an error $E = \alpha E_3 + \beta E_2$

$$|u\rangle = U_{BF}\left[(E|000\rangle) \otimes |00\rangle\right] = U_{BF}\left[(\alpha|100\rangle + \beta|010\rangle) \otimes |00\rangle\right] = \alpha|100\rangle|11\rangle + \beta|010\rangle|10\rangle$$

Now use $V_{BF}$ to directly correct the logical state to its initial value, noting that only one term survives for each term of $|u\rangle$

$$\begin{aligned} V_{BF}|u\rangle = &\left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes \cancel{|10\rangle\langle 10|} + E_3^\dagger \otimes |11\rangle\langle 11|\right) \alpha|100\rangle|11\rangle + \\ &\left(E_0^\dagger \otimes \cancel{|00\rangle\langle 00|} + E_1^\dagger \otimes \cancel{|01\rangle\langle 01|} + E_2^\dagger \otimes |10\rangle\langle 10| + E_3^\dagger \otimes \cancel{|11\rangle\langle 11|}\right) \beta|010\rangle|10\rangle \\ = &\left(\alpha E_3^\dagger|100\rangle\right)|11\rangle + \left(\beta E_2^\dagger|010\rangle\right)|10\rangle = |000\rangle\left(\alpha|11\rangle + \beta|10\rangle\right) \end{aligned}$$

The ancilla can be measured independently and transformed to $|00\rangle$ if they need to be reused

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

$$U_C : |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} (|w_i\rangle \otimes |w_j\rangle)$$

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

$$U_C : |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|w_i\rangle \otimes |w_j\rangle) \quad \longrightarrow \quad |\tilde{\psi}\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|c_i\rangle \otimes |c_j\rangle), \quad |c_i\rangle = U_C|w_i\rangle$$

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

$$U_C : |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|w_i\rangle \otimes |w_j\rangle) \longrightarrow |\tilde{\psi}\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|c_i\rangle \otimes |c_j\rangle), \quad |c_i\rangle = U_C|w_i\rangle$$

Quantum codes must be able to correct errors on such superpositions either in the form of errors on a single block, $E_i \in \mathcal{E}$ or on any number of blocks, $E_{i_1} \otimes \cdots \otimes E_{i_m}$

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

$$U_C : |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|w_i\rangle \otimes |w_j\rangle) \longrightarrow |\tilde{\psi}\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|c_i\rangle \otimes |c_j\rangle), \quad |c_i\rangle = U_C|w_i\rangle$$

Quantum codes must be able to correct errors on such superpositions either in the form of errors on a single block, $E_i \in \mathcal{E}$ or on any number of blocks, $E_{i_1} \otimes \cdots \otimes E_{i_m}$

Suppose the encoded state $|\tilde{\psi}\rangle$ is subjected to error $E_a \otimes E_b$ where both individual errors are correctable for code $C$, apply $U_P \otimes U_P$ for syndrome extraction

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

$$U_C : |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|w_i\rangle \otimes |w_j\rangle) \longrightarrow |\tilde{\psi}\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|c_i\rangle \otimes |c_j\rangle), \quad |c_i\rangle = U_C|w_i\rangle$$

Quantum codes must be able to correct errors on such superpositions either in the form of errors on a single block, $E_i \in \mathcal{E}$ or on any number of blocks, $E_{i_1} \otimes \cdots \otimes E_{i_m}$

Suppose the encoded state $|\tilde{\psi}\rangle$ is subjected to error $E_a \otimes E_b$ where both individual errors are correctable for code $C$, apply $U_P \otimes U_P$ for syndrome extraction

$$U_P \otimes U_P \left[ (E_a \otimes E_b|\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle \right]$$

# Error correction across multiple blocks

A quantum $[[n,k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

$$U_C : |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} (|w_i\rangle \otimes |w_j\rangle) \quad \longrightarrow \quad |\tilde{\psi}\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} (|c_i\rangle \otimes |c_j\rangle), \quad |c_i\rangle = U_C |w_i\rangle$$

Quantum codes must be able to correct errors on such superpositions either in the form of errors on a single block, $E_i \in \mathcal{E}$ or on any number of blocks, $E_{i_1} \otimes \cdots \otimes E_{i_m}$

Suppose the encoded state $|\tilde{\psi}\rangle$ is subjected to error $E_a \otimes E_b$ where both individual errors are correctable for code $C$, apply $U_P \otimes U_P$ for syndrome extraction

$$U_P \otimes U_P \left[ (E_a \otimes E_b |\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ U_P(E_a |c_i\rangle|0\rangle) \otimes U_P(E_b |c_j\rangle|0\rangle) \right]$$

# Error correction across multiple blocks

A quantum $[[n, k]]$ code, $C$ encodes $m \cdot k$ logical qubits in $m \cdot n$ computational qubits

A logical superposition can thus be encoded using an encoding function $U_C$

$$U_C : |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|w_i\rangle \otimes |w_j\rangle) \longrightarrow |\tilde{\psi}\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij}(|c_i\rangle \otimes |c_j\rangle), \quad |c_i\rangle = U_C|w_i\rangle$$

Quantum codes must be able to correct errors on such superpositions either in the form of errors on a single block, $E_i \in \mathcal{E}$ or on any number of blocks, $E_{i_1} \otimes \cdots \otimes E_{i_m}$

Suppose the encoded state $|\tilde{\psi}\rangle$ is subjected to error $E_a \otimes E_b$ where both individual errors are correctable for code $C$, apply $U_P \otimes U_P$ for syndrome extraction

$$U_P \otimes U_P \left[ (E_a \otimes E_b |\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ U_P(E_a|c_i\rangle|0\rangle) \otimes U_P(E_b|c_j\rangle|0\rangle) \right]$$

$$= \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle \right]$$

# Error correction across multiple blocks

$$U_P \otimes U_P \left[ \left( E_a \otimes E_b |\tilde{\psi}\rangle \right) \otimes |0\rangle|0\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a |c_i\rangle|a\rangle \otimes E_b |c_j\rangle|b\rangle \right]$$

# Error correction across multiple blocks

$$U_P \otimes U_P \left[ (E_a \otimes E_b |\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle \right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

# Error correction across multiple blocks

$$U_P \otimes U_P \left[ (E_a \otimes E_b |\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a |c_i\rangle |a\rangle \otimes E_b |c_j\rangle |b\rangle \right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a |c_i\rangle \otimes E_b |c_j\rangle \right]$$

# Error correction across multiple blocks

$$U_P \otimes U_P\left[(E_a \otimes E_b|\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle\right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right]$$

The syndrome $|a\rangle|b\rangle$ indicates that the error can be corrected with ${E_a}^\dagger \otimes {E_b}^\dagger$

# Error correction across multiple blocks

$$U_P \otimes U_P\left[(E_a \otimes E_b|\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle\right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right]$$

The syndrome $|a\rangle|b\rangle$ indicates that the error can be corrected with ${E_a}^\dagger \otimes {E_b}^\dagger$

$${E_a}^\dagger \otimes {E_b}^\dagger|\phi\rangle = {E_a}^\dagger \otimes {E_b}^\dagger \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right]$$

# Error correction across multiple blocks

$$U_P \otimes U_P\left[(E_a \otimes E_b|\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle\right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right]$$

The syndrome $|a\rangle|b\rangle$ indicates that the error can be corrected with ${E_a}^\dagger \otimes {E_b}^\dagger$

$${E_a}^\dagger \otimes {E_b}^\dagger|\phi\rangle = {E_a}^\dagger \otimes {E_b}^\dagger \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[|c_i\rangle \otimes |c_j\rangle\right] = |\tilde{\psi}\rangle$$

# Error correction across multiple blocks

$$U_P \otimes U_P\left[(E_a \otimes E_b|\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle\right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right]$$

The syndrome $|a\rangle|b\rangle$ indicates that the error can be corrected with ${E_a}^\dagger \otimes {E_b}^\dagger$

$${E_a}^\dagger \otimes {E_b}^\dagger|\phi\rangle = {E_a}^\dagger \otimes {E_b}^\dagger \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[|c_i\rangle \otimes |c_j\rangle\right] = |\tilde{\psi}\rangle$$

When using logical states it is important to be able to directly perform computations

# Error correction across multiple blocks

$$U_P \otimes U_P \left[ (E_a \otimes E_b |\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle \right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle \otimes E_b|c_j\rangle \right]$$

The syndrome $|a\rangle|b\rangle$ indicates that the error can be corrected with ${E_a}^\dagger \otimes {E_b}^\dagger$

$${E_a}^\dagger \otimes {E_b}^\dagger |\phi\rangle = {E_a}^\dagger \otimes {E_b}^\dagger \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle \otimes E_b|c_j\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ |c_i\rangle \otimes |c_j\rangle \right] = |\tilde{\psi}\rangle$$

When using logical states it is important to be able to directly perform computations

Let $C$ be a $[[n,k]]$ quantum code with $U_C : W \to C$ encoding function

# Error correction across multiple blocks

$$U_P \otimes U_P \left[ (E_a \otimes E_b |\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle \right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle \otimes E_b|c_j\rangle \right]$$

The syndrome $|a\rangle|b\rangle$ indicates that the error can be corrected with $E_a{}^\dagger \otimes E_b{}^\dagger$

$$E_a{}^\dagger \otimes E_b{}^\dagger |\phi\rangle = E_a{}^\dagger \otimes E_b{}^\dagger \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ E_a|c_i\rangle \otimes E_b|c_j\rangle \right] = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} \left[ |c_i\rangle \otimes |c_j\rangle \right] = |\tilde{\psi}\rangle$$

When using logical states it is important to be able to directly perform computations

Let $C$ be a $[[n, k]]$ quantum code with $U_C : W \to C$ encoding function

In order to compute on the encoded states with a function $U : W \to W$, there must be an analogous unitary operator $\tilde{U}$ acting on the encoded states such that for all $|w\rangle \in W$ sends $U_C|w\rangle$ to $U_C(U|w\rangle)$

# Error correction across multiple blocks

$$U_P \otimes U_P\left[(E_a \otimes E_b|\tilde{\psi}\rangle) \otimes |0\rangle|0\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle|a\rangle \otimes E_b|c_j\rangle|b\rangle\right]$$

Measurement of the two ancilla give $|a\rangle$ and $|b\rangle$ with the computation qubits in state $|\phi\rangle$

$$|\phi\rangle = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right]$$

The syndrome $|a\rangle|b\rangle$ indicates that the error can be corrected with $E_a{}^\dagger \otimes E_b{}^\dagger$

$$E_a{}^\dagger \otimes E_b{}^\dagger|\phi\rangle = E_a{}^\dagger \otimes E_b{}^\dagger \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[E_a|c_i\rangle \otimes E_b|c_j\rangle\right] = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} \alpha_{ij}\left[|c_i\rangle \otimes |c_j\rangle\right] = |\tilde{\psi}\rangle$$

When using logical states it is important to be able to directly perform computations

Let $C$ be a $[[n,k]]$ quantum code with $U_C : W \to C$ encoding function

In order to compute on the encoded states with a function $U : W \to W$, there must be an analogous unitary operator $\tilde{U}$ acting on the encoded states such that for all $|w\rangle \in W$ sends $U_C|w\rangle$ to $U_C(U|w\rangle)$

The operator $\tilde{U} = U_c(U \otimes I)U_C^\dagger$ is one such (inefficient) operator