Today's outline - April 12, 2022

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Reading assignment: 11.2 – 11.3

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Homework Assignment #07:
Chapter 9:2,3,4; Chapter 10:3,4,11
Due Thursday, April 21, 2022

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Thus for any observable O on system A there exists a larger system $X = A \otimes B$, a unitary operator $U : X \to X$, and a state $|\phi\rangle$ such that $S_U^\phi \equiv S_O$

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The discussion on error correction will start by discussing three simple codes which correct single-qubit bit-flip errors, single-qubit phase errors, and all single-qubit errors

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3-qubit states such as $|101\rangle$ are not legitimate logical qubit states which are referred to as codewords

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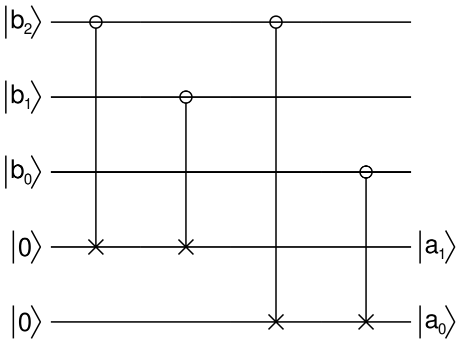

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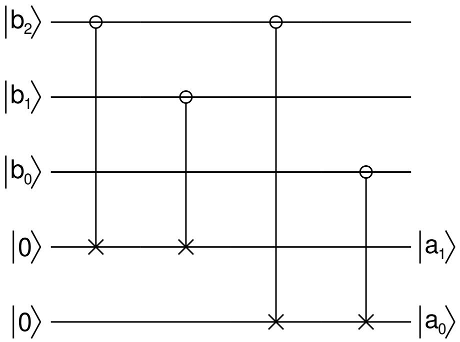
The ancilla qubits are then measured in the standard basis to determine the correction to be applied



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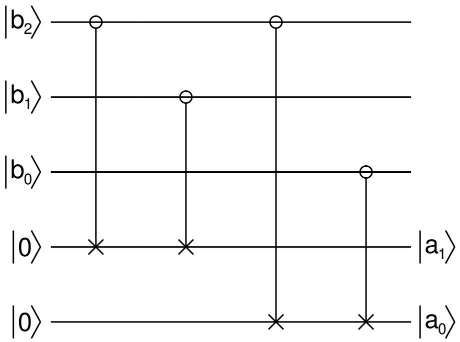

The ancilla qubits are then measured in the standard basis to determine the correction to be applied

The correction circuit must not only repair any bit-flip errors but not corrupt valid codewords

Single-qubit bit-flip correction

Qubit $|a_1\rangle$ will be $|1\rangle$ if $|b_2\rangle \neq |b_1\rangle$ and qubit $|a_0\rangle$ will be $|1\rangle$ if $|b_2\rangle \neq |b_0\rangle$

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Finally, this error correction will only work for a single qubit error, a longer codeword will permit larger number of errors to be corrected

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A general superposition $|\psi\rangle = a|0\rangle + b|1\rangle$ is encoded as

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Since the ancilla register is $|00\rangle$ for both terms no error is detected with this scheme

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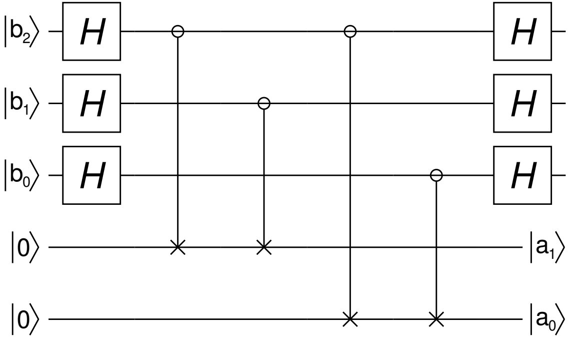

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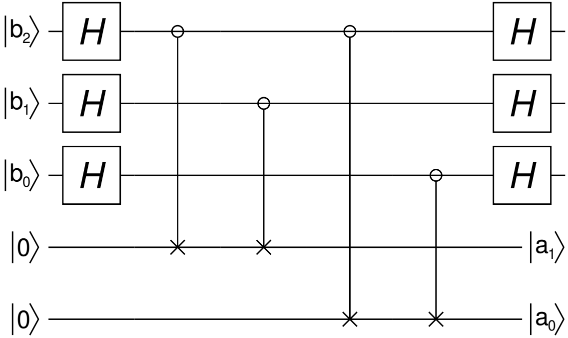

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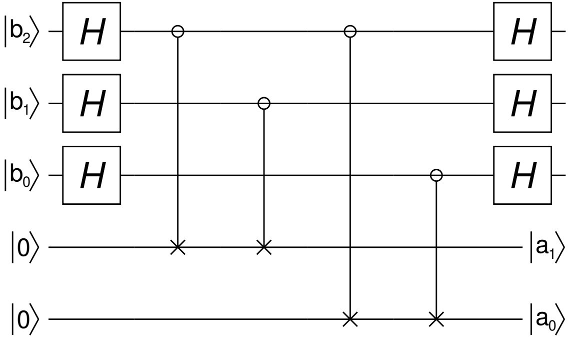

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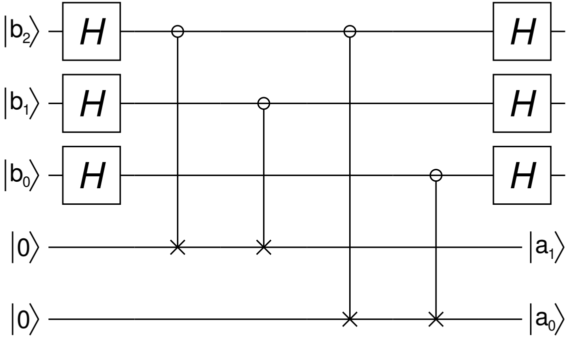

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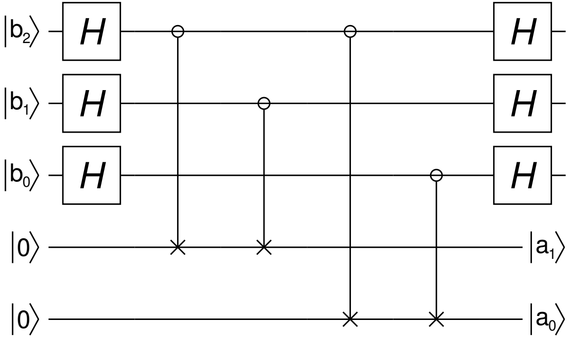

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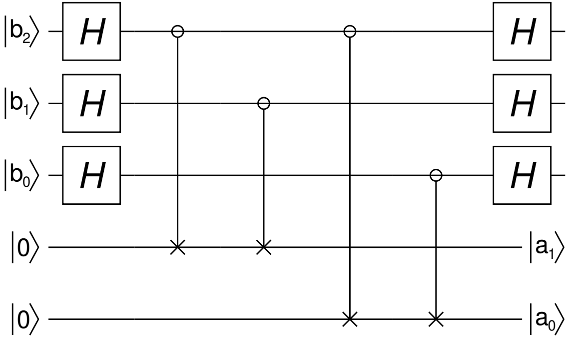

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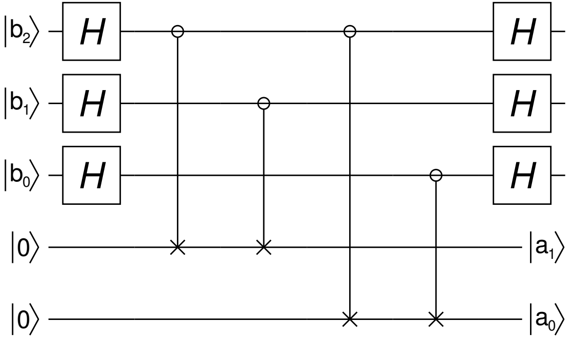

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Error correcting framework

The general theory behind error correcting codes comes from classical error correction

A $[n, k]$ block code C is a 2^k subset of the 2^n possible n-bit strings which form a group, $\mathbf{Z}_2^n$ under bitwise addition modulo 2

The code is linear when the 2^k subset C is a subgroup of $\mathbf{Z}_2^n$

A specific (but non-unique) encoding function $c : \mathbf{Z}_2^k \to \mathbf{Z}_2^n$ is chosen to take the message space (all k-bit strings) into the n-bit code space C

An $m \cdot k$ length message is encoded by separately encoding each of the m k-bit blocks with c to give an $m \cdot n$ length ciphertext

The encoding function, c can be represented by an $n \times k$ generator matrix, G whose k columns form a linearly independent set of binary words

For example for the $[3, 1]$ repetition code we have already seen

$$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example 11.2.2

The $[7, 4]$ Hamming code encodes 4-bit strings, elements of $\mathbf{Z}_2^4$, into 7-bit strings, elements of $\mathbf{Z}_2^7$

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$$1000 \mapsto 1110100$$

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$$1000 \mapsto 1000111$$

The single "1" bit values are mapped to the values of the individual columns

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The other values are just sums of columns modulo 2

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$$\begin{array}{ll} 0001 \mapsto 0001011 & 1001 \mapsto 1001100 \\ 0010 \mapsto 0010101 & 1010 \mapsto 1010010 \\ 0011 \mapsto 0011110 & 1011 \mapsto 1011001 \\ 0100 \mapsto 0100110 & 1100 \mapsto 1100001 \\ 0101 \mapsto 0101100 & 1101 \mapsto 1101010 \\ 0110 \mapsto 0110011 & 1110 \mapsto 1110100 \\ 0111 \mapsto 0111001 & 1111 \mapsto 1111111 \\ 1000 \mapsto 1000111 & 0000 \mapsto 0000000 \end{array}$$

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All m zeroes always maps to n zeroes