Today's outline - April 07, 2022

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Reading assignment: 11.1 – 11.2

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Reading assignment: 11.1 – 11.2

Quantum circuit simulator https://algassert.com/quirk

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However for a general unitary operator, it is not possible to deduce ρ'_A from ρ_A and U alone as ρ'_A depends on the initial state $|\psi\rangle$ of the entire system

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$$S : \rho \mapsto \sum_i p_i S(\rho_i)$$

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Let $X = A \otimes B$, where A and B are single qubit systems

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Example 10.4.1 (cont.)

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$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

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Example 10.4.1 (cont.)

$$U = |00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

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Consider the operator $U_{switch} = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$ acting on single qubit systems A and B

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Note that U_{switch} is not reversible

Operator sum decomposition

In general superoperators are not reversible, of the form $U\rho U^\dagger$ where U is unitary or even of the form $A\rho A^\dagger$ where A is a linear operator

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Each term in the operator sum decomposition is Hermitian and positive but does not necessarily have trace one

$$\mathrm{Tr}(A_i \rho A_i^\dagger) \geq 0$$

However, a density operator, ρ_{decomp}, can be constructed by normalizing

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$$\rho' = \sum_{j=0}^{K-1} p_j \frac{P_j \rho P_j^\dagger}{\text{Tr}(P_j \rho P_j^\dagger)}$$

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If A_i is the operator obtained in the operator sum decomposition for S_U^ϕ when using basis $\{|\beta_i\rangle\}$ for the B subsystem

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$$\rho' = \sum_{i=0}^{K-1} p_i \rho_i, \qquad \rho_i = \mathrm{Tr}_B \left(\frac{(I \otimes P_i) U (\rho \otimes |\phi\rangle\langle\phi|) U^\dagger (I \otimes P_i^\dagger)}{\mathrm{Tr}\left((I \otimes P_i) U (\rho \otimes |\phi\rangle\langle\phi|) U^\dagger (I \otimes P_i^\dagger) \right)} \right)$$

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So the density operator $\rho' = \sum_i p_i\rho_i \equiv S_U^\phi(\rho)$

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Find the operator sum decomposition for C_{not} and $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

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$$\begin{aligned}
A_0|\psi\rangle &= \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle \\
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&= \big(\langle 0|\langle 0|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\big)|0\rangle \\
&\quad + \big(\langle 1|\langle 0|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\big)|1\rangle
\end{aligned}$$

Example 10.4.3 (cont.)

$$A_0|\psi\rangle = \Big(\langle 0|\langle 0|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle$$
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&= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\langle 0|\phi\rangle|1\rangle = a_0\langle 0|\phi\rangle|0\rangle + a_1\langle 0|\phi\rangle|1\rangle = \langle 0|\phi\rangle|\psi\rangle
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$$+ \Big(\langle 1|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle$$
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Recall that $|\phi\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so

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\end{aligned}
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Recall that $|\phi\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so $\qquad A_0|\psi\rangle = \langle 0|\phi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}|\psi\rangle \quad \longrightarrow \quad A_0 = \frac{1}{\sqrt{2}}I$

Example 10.4.3 (cont.)

$$A_0|\psi\rangle = \Big(\langle 0|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle$$

$$+ \Big(\langle 1|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle$$

$$= \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle$$

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\begin{aligned}
A_0|\psi\rangle &= \Big(\langle 0|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle \\
&\quad + \Big(\langle 1|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\langle 0|\phi\rangle|1\rangle = a_0\langle 0|\phi\rangle|0\rangle + a_1\langle 0|\phi\rangle|1\rangle = \langle 0|\phi\rangle|\psi\rangle
\end{aligned}
$$

Recall that $|\phi\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so $\quad A_0|\psi\rangle = \langle 0|\phi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}|\psi\rangle \quad \longrightarrow \quad A_0 = \frac{1}{\sqrt{2}} I$

Similarly for A_1 we have

$$
\begin{aligned}
A_1|\psi\rangle &= \langle 0|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|X|\psi\rangle\langle 1|\phi\rangle|0\rangle + \langle 1|X|\psi\rangle\langle 1|\phi\rangle|1\rangle
\end{aligned}
$$

Example 10.4.3 (cont.)

$$
\begin{aligned}
A_0|\psi\rangle &= \big(\langle 0|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\big)|0\rangle \\
&\quad + \big(\langle 1|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\big)|1\rangle \\
&= \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\big)|0\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\big)|1\rangle \\
&= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\langle 0|\phi\rangle|1\rangle = a_0\langle 0|\phi\rangle|0\rangle + a_1\langle 0|\phi\rangle|1\rangle = \langle 0|\phi\rangle|\psi\rangle
\end{aligned}
$$

Recall that $|\phi\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so

$$A_0|\psi\rangle = \langle 0|\phi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}|\psi\rangle \quad \longrightarrow \quad A_0 = \frac{1}{\sqrt{2}}I$$

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A_1|\psi\rangle &= \langle 0|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\big)|0\rangle + \langle 1|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\big)|1\rangle \\
&= \langle 0|X|\psi\rangle\langle 1|\phi\rangle|0\rangle + \langle 1|X|\psi\rangle\langle 1|\phi\rangle|1\rangle = a_1\langle 1|\phi\rangle|0\rangle + a_0\langle 1|\phi\rangle|1\rangle
\end{aligned}
$$

Example 10.4.3 (cont.)

$$A_0|\psi\rangle = \Big(\langle 0|\cancel{\langle 0|}(X\otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I\otimes|0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle$$
$$+\Big(\langle 1|\cancel{\langle 0|}(X\otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I\otimes|0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle$$
$$= \langle 0|\langle 0|(I\otimes|0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 0|(I\otimes|0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle$$
$$= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\langle 0|\phi\rangle|1\rangle = a_0\langle 0|\phi\rangle|0\rangle + a_1\langle 0|\phi\rangle|1\rangle = \langle 0|\phi\rangle|\psi\rangle$$

Recall that $|\phi\rangle\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ so $A_0|\psi\rangle = \langle 0|\phi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}|\psi\rangle \longrightarrow A_0 = \frac{1}{\sqrt{2}}I$

Similarly for A_1 we have

$$A_1|\psi\rangle = \langle 0|\langle 1|(X\otimes|1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 1|(X\otimes|1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|1\rangle$$
$$= \langle 0|X|\psi\rangle\langle 1|\phi\rangle|0\rangle + \langle 1|X|\psi\rangle\langle 1|\phi\rangle|1\rangle = a_1\langle 1|\phi\rangle|0\rangle + a_0\langle 1|\phi\rangle|1\rangle = \langle 1|\phi\rangle X|\psi\rangle$$

Example 10.4.3 (cont.)

$$\begin{aligned}
A_0|\psi\rangle &= \Big(\langle 0|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle \\
&\quad + \Big(\langle 1|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\langle 0|\phi\rangle|1\rangle = a_0\langle 0|\phi\rangle|0\rangle + a_1\langle 0|\phi\rangle|1\rangle = \langle 0|\phi\rangle|\psi\rangle
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Recall that $|\phi\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so $\quad A_0|\psi\rangle = \langle 0|\phi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}|\psi\rangle \quad\longrightarrow\quad A_0 = \frac{1}{\sqrt{2}}I$

Similarly for A_1 we have

$$\begin{aligned}
A_1|\psi\rangle &= \langle 0|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|X|\psi\rangle\langle 1|\phi\rangle|0\rangle + \langle 1|X|\psi\rangle\langle 1|\phi\rangle|1\rangle = a_1\langle 1|\phi\rangle|0\rangle + a_0\langle 1|\phi\rangle|1\rangle = \langle 1|\phi\rangle X|\psi\rangle \\
&= \frac{1}{\sqrt{2}}X|\psi\rangle
\end{aligned}$$

Example 10.4.3 (cont.)

$$
\begin{aligned}
A_0|\psi\rangle &= \Big(\langle 0|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle \\
&\quad + \Big(\langle 1|\cancel{\langle 0|}(X \otimes \cancel{|1\rangle}\langle 1|)|\psi\rangle|\phi\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 0|(I \otimes |0\rangle\langle 0|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\langle 0|\phi\rangle|1\rangle = a_0\langle 0|\phi\rangle|0\rangle + a_1\langle 0|\phi\rangle|1\rangle = \langle 0|\phi\rangle|\psi\rangle
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Recall that $|\phi\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so

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Similarly for A_1 we have

$$
\begin{aligned}
A_1|\psi\rangle &= \langle 0|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|0\rangle + \langle 1|\langle 1|(X \otimes |1\rangle\langle 1|)|\psi\rangle|\phi\rangle\Big)|1\rangle \\
&= \langle 0|X|\psi\rangle\langle 1|\phi\rangle|0\rangle + \langle 1|X|\psi\rangle\langle 1|\phi\rangle|1\rangle = a_1\langle 1|\phi\rangle|0\rangle + a_0\langle 1|\phi\rangle|1\rangle = \langle 1|\phi\rangle X|\psi\rangle \\
&= \frac{1}{\sqrt{2}}X|\psi\rangle \qquad \longrightarrow \qquad A_1 = \frac{1}{\sqrt{2}}X
\end{aligned}
$$

Example 10.4.4

Find the operator sum decomposition for U_{switch} and $|\phi\rangle = |0\rangle$

$$U_{switch} = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$$

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Find the operator sum decomposition for U_{switch} and $|\phi\rangle = |0\rangle$

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$$S_U^{\phi} = \mathrm{Tr}_B \left(U(\rho \otimes |\phi\rangle\langle\phi|)U^{\dagger} \right)$$

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$$A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle$$

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$$A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 00|(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|)|\psi\phi\rangle|0\rangle$$
$$+ \langle 10|(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|)|\psi\phi\rangle|1\rangle$$

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$$A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 00|\left(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle$$

$$+ \langle 10|\left(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|1\rangle$$

$$= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle$$

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Find the operator sum decomposition for U_{switch} and $|\phi\rangle = |0\rangle$

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$$\begin{aligned} A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 00|\left(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle \\ &\quad + \langle 10|\left(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|1\rangle \\ &= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle = \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\langle 1|\phi\rangle|1\rangle \end{aligned}$$

Example 10.4.4

Find the operator sum decomposition for U_{switch} and $|\phi\rangle = |0\rangle$

$$U_{switch} = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$$

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$$\begin{aligned} A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 00|\left(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle \\ &\quad + \langle 10|\left(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|1\rangle \\ &= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle = \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle \end{aligned}$$

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$$A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 00|(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|})|\psi\phi\rangle|0\rangle$$

$$+ \langle 10|(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|})|\psi\phi\rangle|1\rangle$$

$$= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle = \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 0|\psi\rangle$$

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Find the operator sum decomposition for U_{switch} and $|\phi\rangle = |0\rangle$

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$$A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 00|\left(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle$$

$$+ \langle 10|\left(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|1\rangle$$

$$= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle = \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 0|\psi\rangle \longrightarrow A_0 = |0\rangle\langle 0|$$

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Example 10.4.4

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$$A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 00|\big(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\big)|\psi\phi\rangle|0\rangle$$
$$+ \langle 10|\big(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\big)|\psi\phi\rangle|1\rangle$$
$$= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle = \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 0|\psi\rangle \longrightarrow A_0 = |0\rangle\langle 0|$$

$$A_1|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 1|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 01|\big(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|\big)|\psi\phi\rangle|0\rangle$$
$$+ \langle 11|\big(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|\big)|\psi\phi\rangle|1\rangle$$

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$$+ \langle 10|\big(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\big)|\psi\phi\rangle|1\rangle$$
$$= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle = \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 0|\psi\rangle \longrightarrow A_0 = |0\rangle\langle 0|$$

$$A_1|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 1|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 01|\big(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + |01\rangle\langle 10| + \cancel{|11\rangle\langle 11|}\big)|\psi\phi\rangle|0\rangle$$
$$+ \langle 11|\big(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + |11\rangle\langle 11|\big)|\psi\phi\rangle|1\rangle$$

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$$S_U^{\phi} = \mathrm{Tr}_B\left(U(\rho \otimes |\phi\rangle\langle\phi|)U^{\dagger}\right) = A_0\rho A_0^{\dagger} + A_1\rho A_1^{\dagger}, \qquad A_i = \langle i|U|\phi\rangle$$

$$\begin{aligned} A_0|\psi\rangle = \sum_{i=0}^{1}\langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 00|(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|})|\psi\phi\rangle|0\rangle \\ &\quad + \langle 10|(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|})|\psi\phi\rangle|1\rangle \\ = \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle &= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 0|\psi\rangle \longrightarrow A_0 = |0\rangle\langle 0| \end{aligned}$$

$$\begin{aligned} A_1|\psi\rangle = \sum_{i=0}^{1}\langle\alpha_i|\langle 1|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 01|(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + |01\rangle\langle 10| + \cancel{|11\rangle\langle 11|})|\psi\phi\rangle|0\rangle \\ &\quad + \langle 11|(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + |11\rangle\langle 11|)|\psi\phi\rangle|1\rangle \\ = \langle 10|\psi\phi\rangle|0\rangle + \langle 11|\psi\phi\rangle|1\rangle & \end{aligned}$$

Example 10.4.4

Find the operator sum decomposition for U_{switch} and $|\phi\rangle = |0\rangle$

$$U_{switch} = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$$

$$S_U^\phi = \text{Tr}_B\left(U(\rho\otimes|\phi\rangle\langle\phi|)U^\dagger\right) = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger, \qquad A_i = \langle i|U|\phi\rangle$$

$$A_0|\psi\rangle = \sum_{i=0}^{1}\langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 00|\left(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle$$
$$+ \langle 10|\left(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|1\rangle$$
$$= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle = \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 0|\psi\rangle \longrightarrow A_0 = |0\rangle\langle 0|$$

$$A_1|\psi\rangle = \sum_{i=0}^{1}\langle\alpha_i|\langle 1|U|\psi\rangle|\phi\rangle|\alpha_i\rangle = \langle 01|\left(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + |01\rangle\langle 10| + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle$$
$$+ \langle 11|\left(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + |11\rangle\langle 11|\right)|\psi\phi\rangle|1\rangle$$
$$= \langle 10|\psi\phi\rangle|0\rangle + \langle 11|\psi\phi\rangle|1\rangle = \langle 1|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\langle 1|\phi\rangle|1\rangle$$

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$$\begin{aligned}
A_0|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 0|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 00|\left(|00\rangle\langle 00| + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle \\
&\quad + \langle 10|\left(\cancel{|00\rangle\langle 00|} + |10\rangle\langle 01| + \cancel{|01\rangle\langle 10|} + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|1\rangle \\
= \langle 00|\psi\phi\rangle|0\rangle + \langle 01|\psi\phi\rangle|1\rangle &= \langle 0|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 0|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 0|\psi\rangle \longrightarrow A_0 = |0\rangle\langle 0|
\end{aligned}$$

$$\begin{aligned}
A_1|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 1|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 01|\left(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + |01\rangle\langle 10| + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle \\
&\quad + \langle 11|\left(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + |11\rangle\langle 11|\right)|\psi\phi\rangle|1\rangle \\
= \langle 10|\psi\phi\rangle|0\rangle + \langle 11|\psi\phi\rangle|1\rangle &= \langle 1|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle
\end{aligned}$$

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$$\begin{aligned} A_1|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 1|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 01|\big(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + |01\rangle\langle 10| + \cancel{|11\rangle\langle 11|}\big)|\psi\phi\rangle|0\rangle \\ &\quad + \langle 11|\big(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + |11\rangle\langle 11|\big)|\psi\phi\rangle|1\rangle \\ &= \langle 10|\psi\phi\rangle|0\rangle + \langle 11|\psi\phi\rangle|1\rangle = \langle 1|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 1|\psi\rangle \end{aligned}$$

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Find the operator sum decomposition for U_{switch} and $|\phi\rangle = |0\rangle$

$$U_{switch} = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$$

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$$\begin{aligned} A_1|\psi\rangle = \sum_{i=0}^{1} \langle\alpha_i|\langle 1|U|\psi\rangle|\phi\rangle|\alpha_i\rangle &= \langle 01|\left(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + |01\rangle\langle 10| + \cancel{|11\rangle\langle 11|}\right)|\psi\phi\rangle|0\rangle \\ &+ \langle 11|\left(\cancel{|00\rangle\langle 00|} + \cancel{|10\rangle\langle 01|} + \cancel{|01\rangle\langle 10|} + |11\rangle\langle 11|\right)|\psi\phi\rangle|1\rangle \\ = \langle 10|\psi\phi\rangle|0\rangle + \langle 11|\psi\phi\rangle|1\rangle &= \langle 1|\psi\rangle\langle 0|\phi\rangle|0\rangle + \langle 1|\psi\rangle\cancel{\langle 1|\phi\rangle}|1\rangle = |0\rangle\langle 1|\psi\rangle \longrightarrow A_1 = |0\rangle\langle 1| \end{aligned}$$