Today's outline - March 31, 2022

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Reading assignment: 10.2 – 10.3

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Homework Assignment #06:
See Blackboard
Due Tuesday, April 05, 2022

Quantum circuit simulator https://algassert.com/quirk

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Let B be a quantum system with a vector space of dimension $2^n > M$ and let $\{|\phi_0\rangle, \dots, |\phi_{M-1}\rangle\}$ be the first M elements of an orthonormal basis for B, define

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The density operator of a pure state is a projection operator, such that $\rho_x^X \rho_x^X = \rho_x^X$

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$$\det(\rho) = \begin{vmatrix} \frac{1+z}{2} & \frac{x-iy}{2} \\ \frac{x+iy}{2} & \frac{1-z}{2} \end{vmatrix}$$

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Geometry of mixed states

It is possible to use the Bloch sphere to visualize single-qubit mixed states which are linear combinations of pure states with non-negative coefficients that sum to 1

A density operator for a single qubit must be Hermitian and self-adjoint with trace 1 and the general form

$$\rho = \begin{pmatrix} a & c - id \\ c + id & b \end{pmatrix}$$

The condition on the trace means that the matrix can be written with only 3 real parameters: x, y, z and expanded in terms of Pauli matrices

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A maximally uncertain state for an n-qubit system has all the diagonal elements equal to 2^{-n} so $S(\rho) = n$

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The von Neumann entropy for a single qubit system is just a function of the distance of the state from the center of the Bloch sphere

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The 2-qubit system is the simplest bipartite system with a maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

For each of the two qubits, the density matrix ρ_{ME} has maximal von Neumann entropy

$$\rho_{ME} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S(\rho) = -2\left[\tfrac{1}{2}\log_2\left(\tfrac{1}{2}\right)\right] = 2$$

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It makes sense to use the von Neumann entropy of the partial trace as a measure of the entanglement if it can be assumed that the partial trace is the same for each of the two subsystems, A and B

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The amount of entanglement between the two parts of a pure state $|\psi\rangle \in X = A \otimes B$ with density operator $\rho = |\psi\rangle\langle\psi|$ is defined to be $S(\text{Tr}_A(\rho))$ or $S(\text{Tr}_B(\rho))$

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$$\left(\rho_y^A\right)_{01} = \tfrac{1}{2}(\cancel{\langle 00|01\rangle} + \cancel{\langle 00|10\rangle})(\cancel{\langle 01|10\rangle} + \langle 10|10\rangle) + \tfrac{1}{2}(\langle 01|01\rangle + \cancel{\langle 01|10\rangle})(\cancel{\langle 01|11\rangle} + \cancel{\langle 10|11\rangle})$$
$$= 0$$

$$\left(\rho_y^A\right)_{10} = \tfrac{1}{2}(\cancel{\langle 10|01\rangle} + \langle 10|10\rangle)(\cancel{\langle 01|00\rangle} + \cancel{\langle 10|00\rangle}) + \tfrac{1}{2}(\cancel{\langle 11|01\rangle} + \cancel{\langle 11|10\rangle})(\langle 01|01\rangle + \cancel{\langle 10|01\rangle})$$
$$= 0$$

$$\left(\rho_y^A\right)_{11} = \tfrac{1}{2}(\langle 10|01\rangle + \langle 10|10\rangle)(\langle 01|10\rangle + \langle 10|10\rangle) + \tfrac{1}{2}(\langle 11|01\rangle + \langle 11|10\rangle)(\langle 01|11\rangle + \langle 10|11\rangle)$$

Example 10.2.1 (cont.)

$$
\begin{aligned}
\left(\rho_y^A\right)_{01} &= \tfrac{1}{2}\left(\cancel{\langle 00|01\rangle} + \cancel{\langle 00|10\rangle}\right)\left(\cancel{\langle 01|10\rangle} + \langle 10|10\rangle\right) + \tfrac{1}{2}\left(\langle 01|01\rangle + \cancel{\langle 01|10\rangle}\right)\left(\cancel{\langle 01|11\rangle} + \cancel{\langle 10|11\rangle}\right) \\
&= 0 \\
\left(\rho_y^A\right)_{10} &= \tfrac{1}{2}\left(\cancel{\langle 10|01\rangle} + \langle 10|10\rangle\right)\left(\cancel{\langle 01|00\rangle} + \cancel{\langle 10|00\rangle}\right) + \tfrac{1}{2}\left(\cancel{\langle 11|01\rangle} + \cancel{\langle 11|10\rangle}\right)\left(\langle 01|01\rangle + \cancel{\langle 10|01\rangle}\right) \\
&= 0 \\
\left(\rho_y^A\right)_{11} &= \tfrac{1}{2}\left(\cancel{\langle 10|01\rangle} + \langle 10|10\rangle\right)\left(\cancel{\langle 01|10\rangle} + \langle 10|10\rangle\right) + \tfrac{1}{2}\left(\cancel{\langle 11|01\rangle} + \cancel{\langle 11|10\rangle}\right)\left(\cancel{\langle 01|11\rangle} + \cancel{\langle 10|11\rangle}\right) \\
&= \tfrac{1}{2}
\end{aligned}
$$

Example 10.2.1 (cont.)

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(\rho_y^A)_{01} &= \tfrac{1}{2}(\cancel{\langle 00|01\rangle} + \cancel{\langle 00|10\rangle})(\cancel{\langle 01|10\rangle} + \langle 10|10\rangle) + \tfrac{1}{2}(\langle 01|01\rangle + \cancel{\langle 01|10\rangle})(\cancel{\langle 01|11\rangle} + \cancel{\langle 10|11\rangle}) \\
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\end{aligned}$$

Thus $\rho_y^A = \frac{1}{2}I = \rho_y^B = \rho_{ME}$ and the entropy, $S(\rho_{ME}) = 1$ for this state also

Example 10.2.1 (cont.)

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Thus $\rho_y^A = \frac{1}{2}I = \rho_y^B = \rho_{ME}$ and the entropy, $S(\rho_{ME}) = 1$ for this state also

Any other 2-qubit maximally entangled state will give the same results

Example 10-2-2

Compute the partial density operator for the first qubit of the state

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$$|x\rangle = \frac{7}{10}|00\rangle + \frac{1}{10}|01\rangle + \frac{1}{10}|10\rangle + \frac{7}{10}|11\rangle$$

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$$S(\rho_x^A) = -\tfrac{16}{25}\log_2\tfrac{16}{25} - \tfrac{9}{25}\log_2\tfrac{9}{25} = 0.942$$

Example 10.2.4

Determine the amount of entanglement in the 4-qubit state in the 2,4 and 1,2 subsystems

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$$|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle) = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$$

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$$\rho_\psi^{1,2} = \mathrm{Tr}_{3,4}(|\psi\rangle\langle\psi|) = \sum_{i,j=0}^{3}\sum_{k=0}^{3}\langle j_3|\langle k_4||\psi\rangle\langle\psi||i_3\rangle|k_4\rangle|j\rangle\langle i|$$

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Determine the amount of entanglement in the 4-qubit state in the 2,4 and 1,2 subsystems

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle) = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$$

In the 2,4 decomposition, this state is unentangled

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_3 + |1\rangle_1|1\rangle_3) \otimes \frac{1}{\sqrt{2}}(|0\rangle_2|0\rangle_4 + |1\rangle_2|1\rangle_4)$$

Since the state is unentangled, it is a pure state in the 2,4 subsystem and $S(\rho_\psi^{2,4}) \equiv 0$

In the 1,2 and 3,4 decomposition, the partial density operator becomes

$$\rho_\psi^{1,2} = \mathrm{Tr}_{3,4}(|\psi\rangle\langle\psi|) = \sum_{i,j=0}^{3}\sum_{k=0}^{3}\langle j_3|\langle k_4||\psi\rangle\langle\psi||i_3\rangle|k_4\rangle|j\rangle\langle i|$$

The coefficient of $|j\rangle\langle i|$ is $\frac{1}{4}\delta_{ij}$ so

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In this decomposition the state is maximally entangled

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