Today's outline - March 29, 2022

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Reading assignment: 10.2 – 10.3

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Homework Assignment #06:
See Blackboard
Due Tuesday, April 05, 2022

Quantum circuit simulator https://algassert.com/quirk

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When all these systems are measured, the results are the same as for the measurement of the pure states but the system is fundamentally different since pure states have phase information that can produce interference effects not found in mixed states

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Knowing the mixed states of all the subsystems only provides full knowledge of the system when it is unentangled in that specific subspace decomposition

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This is a useful method of computing an inner product

Restricting operators to subsystems

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All information from subsystem A alone can be obtained with the density operator

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$$\rho_x^A = \mathrm{Tr}_B(\rho_x^X) = \mathrm{Tr}_B\left(\sum_{i=0}^{M-1}\sum_{j=0}^{L-1}\sum_{k=0}^{M-1}\sum_{l=0}^{L-1}\overline{x_{ij}}x_{kl}|\alpha_k\rangle|\beta_l\rangle\langle\alpha_i|\langle\beta_j|\right)$$

$$= \sum_{u=0}^{M-1}\sum_{v=0}^{M-1}\left[\sum_{w=0}^{L-1}\langle\alpha_u|\langle\beta_w|\left(\sum_{i=0}^{M-1}\sum_{j=0}^{L-1}\sum_{k=0}^{M-1}\sum_{l=0}^{L-1}\overline{x_{ij}}x_{kl}|\alpha_k\rangle|\beta_l\rangle\langle\alpha_i|\langle\beta_j|\right)|\alpha_v\rangle|\beta_w\rangle\right]|\alpha_u\rangle\langle\alpha_v|$$

$$= \sum_{u=0}^{M-1}\sum_{v=0}^{M-1}\sum_{w=0}^{L-1}\overline{x_{vw}}x_{uw}|\alpha_u\rangle\langle\alpha_v|$$

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However, it is not possible to recover the state of the entire system from the set of all subsystem density operators

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Example 10.1.1 (cont.)

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$$\begin{aligned}
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The density operators for the individual qubits subsystems are

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$$\rho_\psi^A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_\psi^B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$