Today's outline - March 24, 2022

Today's outline - March 24, 2022

- Improving Grover's algorithm

Today's outline - March 24, 2022

- Improving Grover's algorithm
  (a) Modifying the iteration distance

Today's outline - March 24, 2022

- Improving Grover's algorithm
  (a) Modifying the iteration distance
  (b) Modifying the last iteration

Today's outline - March 24, 2022

- Improving Grover's algorithm
  (a) Modifying the iteration distance
  (b) Modifying the last iteration
- Solving for t

Today's outline - March 24, 2022

- Improving Grover's algorithm
  (a) Modifying the iteration distance
  (b) Modifying the last iteration
- Solving for t
  (a) Repeated random iterations

Today's outline - March 24, 2022

- Improving Grover's algorithm
  (a) Modifying the iteration distance
  (b) Modifying the last iteration
- Solving for t
  (a) Repeated random iterations
  (b) Quantum counting

Today's outline - March 24, 2022

- Improving Grover's algorithm
  (a) Modifying the iteration distance
  (b) Modifying the last iteration
- Solving for t
  (a) Repeated random iterations
  (b) Quantum counting

Reading assignment: 10.1 – 10.3

Today's outline - March 24, 2022

- Improving Grover's algorithm
  (a) Modifying the iteration distance
  (b) Modifying the last iteration
- Solving for t
  (a) Repeated random iterations
  (b) Quantum counting

Reading assignment: 10.1 – 10.3

Homework Assignment #06:
See Blackboard
Due Thursday, March 31, 2022

Quantum circuit simulator https://algassert.com/quirk

Improving Grover's algorithm

When I chose the angles for the geometric argument, I selected $g_0 = \sin\left(\frac{\pi}{18}\right)$

Improving Grover's algorithm

When I chose the angles for the geometric argument, I selected $g_0 = \sin\left(\frac{\pi}{18}\right)$

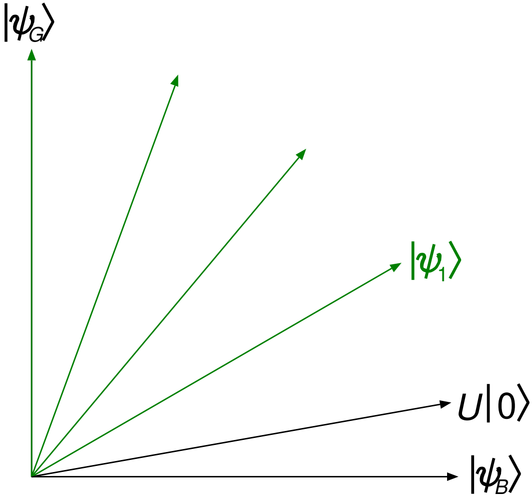
This leads to an optimal case where we can obtain $g_i = 1$ as can be seen from the diagram

Improving Grover's algorithm

When I chose the angles for the geometric argument, I selected $g_0 = \sin\left(\frac{\pi}{18}\right)$

This leads to an optimal case where we can obtain $g_i = 1$ as can be seen from the diagram

Grover's algorithm is not inherently probabilstic as evident from the geometric discussion



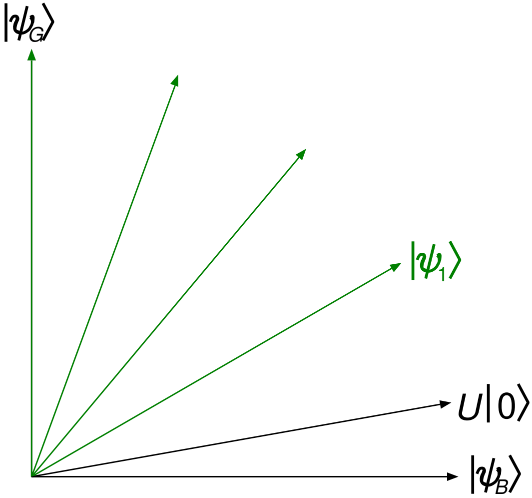
Improving Grover's algorithm

When I chose the angles for the geometric argument, I selected $g_0 = \sin\left(\frac{\pi}{18}\right)$

This leads to an optimal case where we can obtain $g_i = 1$ as can be seen from the diagram

Grover's algorithm is not inherently probabilstic as evident from the geometric discussion

Thus it is possible to improve on it and guarantee a solution with quadratic speedup



Improving Grover's algorithm

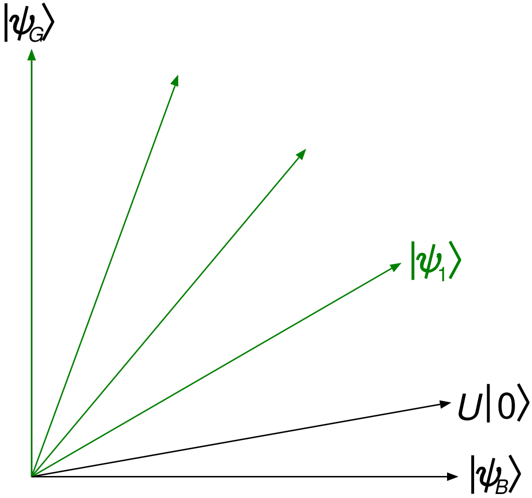
When I chose the angles for the geometric argument, I selected $g_0 = \sin\left(\frac{\pi}{18}\right)$

This leads to an optimal case where we can obtain $g_i = 1$ as can be seen from the diagram

Grover's algorithm is not inherently probabilstic as evident from the geometric discussion

Thus it is possible to improve on it and guarantee a solution with quadratic speedup

There are two different methods to accomplish this



Modifying each step

Recall that the probability t is related to an angle θ

Modifying each step

Recall that the probability t is related to an angle θ $\sqrt{t} = \sin\theta = g_0$

Modifying each step

Recall that the probability t is related to an angle θ $\sqrt{t} = \sin\theta = g_0$

The optimal number of iterations is given by

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0}$$

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Thus the strategy to getting the solution with certainty is to replace U by U' with success probability $g_0' < g_0$ such that for $\sin\theta' = g_0'$, $\frac{\pi}{4\theta'} - \frac{1}{2}$ is an integer

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Thus the strategy to getting the solution with certainty is to replace U by U' with success probability $g_0' < g_0$ such that for $\sin\theta' = g_0'$, $\frac{\pi}{4\theta'} - \frac{1}{2}$ is an integer

This can be done by using an additional qubit b

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Thus the strategy to getting the solution with certainty is to replace U by U' with success probability $g_0' < g_0$ such that for $\sin\theta' = g_0'$, $\frac{\pi}{4\theta'} - \frac{1}{2}$ is an integer

This can be done by using an additional qubit b

Given U with success probability g_0 which acts on an n-qubit register $|s\rangle$, define $U' = U \otimes B$ acting on an $(n+1)$-qubit register $|s\rangle|b\rangle$ where

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Thus the strategy to getting the solution with certainty is to replace U by U' with success probability $g_0' < g_0$ such that for $\sin\theta' = g_0'$, $\frac{\pi}{4\theta'} - \frac{1}{2}$ is an integer

This can be done by using an additional qubit b

Given U with success probability g_0 which acts on an n-qubit register $|s\rangle$, define $U' = U \otimes B$ acting on an $(n+1)$-qubit register $|s\rangle|b\rangle$ where

$$B|0\rangle = \sqrt{1 - \frac{{g_0'}^2}{g_0^2}}|0\rangle + \sqrt{\frac{{g_0'}^2}{g_0^2}}|1\rangle$$

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Thus the strategy to getting the solution with certainty is to replace U by U' with success probability $g_0' < g_0$ such that for $\sin\theta' = g_0'$, $\frac{\pi}{4\theta'} - \frac{1}{2}$ is an integer

This can be done by using an additional qubit b

Given U with success probability g_0 which acts on an n-qubit register $|s\rangle$, define $U' = U \otimes B$ acting on an $(n+1)$-qubit register $|s\rangle|b\rangle$ where

$$B|0\rangle = \sqrt{1 - \frac{{g_0'}^2}{g_0^2}}|0\rangle + \sqrt{\frac{{g_0'}^2}{g_0^2}}|1\rangle$$

Amplitude amplification is performed on the $(n+1)$-qubit state using an iteration operator

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Thus the strategy to getting the solution with certainty is to replace U by U' with success probability $g_0' < g_0$ such that for $\sin\theta' = g_0'$, $\frac{\pi}{4\theta'} - \frac{1}{2}$ is an integer

This can be done by using an additional qubit b

Given U with success probability g_0 which acts on an n-qubit register $|s\rangle$, define $U' = U \otimes B$ acting on an $(n+1)$-qubit register $|s\rangle|b\rangle$ where

$$B|0\rangle = \sqrt{1 - \frac{{g_0'}^2}{g_0^2}}|0\rangle + \sqrt{\frac{{g_0'}^2}{g_0^2}}|1\rangle$$

Amplitude amplification is performed on the $(n+1)$-qubit state using an iteration operator

$$Q' = -U' S_0^{\pi} (U')^{-1} S_{G'}^{\pi}$$

Modifying each step

Recall that the probability t is related to an angle θ

$$\sqrt{t} = \sin\theta = g_0$$

The optimal number of iterations is given by

$$i \approx \frac{\pi}{4g_0} \approx \frac{\pi}{4\theta}$$

If the quantity $\frac{\pi}{4\theta} - \frac{1}{2}$ is an integer then for

$$i = \frac{\pi}{4\theta} - \frac{1}{2} \quad \longrightarrow \quad g_i \equiv 1$$

Thus the strategy to getting the solution with certainty is to replace U by U' with success probability $g_0' < g_0$ such that for $\sin\theta' = g_0'$, $\frac{\pi}{4\theta'} - \frac{1}{2}$ is an integer

This can be done by using an additional qubit b

Given U with success probability g_0 which acts on an n-qubit register $|s\rangle$, define $U' = U \otimes B$ acting on an $(n+1)$-qubit register $|s\rangle|b\rangle$ where

$$B|0\rangle = \sqrt{1 - \frac{{g_0'}^2}{g_0^2}}|0\rangle + \sqrt{\frac{{g_0'}^2}{g_0^2}}|1\rangle$$

Amplitude amplification is performed on the $(n+1)$-qubit state using an iteration operator

$$Q' = -U' S_0^{\pi} (U')^{-1} S_{G'}^{\pi}$$

This algorithm now succeeds with certainty in $i = \frac{\pi}{4\theta'} - \frac{1}{2}$ steps

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

$$Q(\phi, \tau) = -US_0^{\phi} U^{-1} S_G^{\tau}$$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

$$Q(\phi, \tau) = -US_0^{\phi} U^{-1} S_G^{\tau}$$

$$S_X^{\phi} = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

$$Q(\phi, \tau) = -US_0^{\phi} U^{-1} S_G^{\tau}$$

S_X^{ϕ} can be implemented efficiently using an ancilla qubit as shown previously

$$S_X^{\phi} = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

$$Q(\phi, \tau) = -US_0^{\phi}U^{-1}S_G^{\tau}$$

S_X^{ϕ} can be implemented efficiently using an ancilla qubit as shown previously

$$S_X^{\phi} = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

Given any quantum state $|v\rangle$, we can write the general form of the relation obtained for $\phi = \pi$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

S_X^ϕ can be implemented efficiently using an ancilla qubit as shown previously

Given any quantum state $|v\rangle$, we can write the general form of the relation obtained for $\phi = \pi$

$$Q(\phi, \tau) = -US_0^\phi U^{-1} S_G^\tau$$

$$S_X^\phi = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

$$US_0^\phi U^{-1}|v\rangle = |v\rangle - \left(1 - e^{i\phi}\right) \overline{\langle v|U|v\rangle} U|0\rangle$$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

$$Q(\phi,\tau) = -US_0^{\phi}U^{-1}S_G^{\tau}$$

S_X^{ϕ} can be implemented efficiently using an ancilla qubit as shown previously

$$S_X^{\phi} = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

Given any quantum state $|v\rangle$, we can write the general form of the relation obtained for $\phi = \pi$

$$US_0^{\phi}U^{-1}|v\rangle = |v\rangle - \left(1 - e^{i\phi}\right)\overline{\langle v|U|v\rangle}\, U|0\rangle$$

This can be derived by representing $|v\rangle$ in the $U|i\rangle$ basis and applying $US_0^{\phi}U^{-1}$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

S_X^ϕ can be implemented efficiently using an ancilla qubit as shown previously

Given any quantum state $|v\rangle$, we can write the general form of the relation obtained for $\phi = \pi$

This can be derived by representing $|v\rangle$ in the $U|i\rangle$ basis and applying $US_0^\phi U^{-1}$

$$Q(\phi, \tau) = -US_0^\phi U^{-1} S_G^\tau$$

$$S_X^\phi = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

$$US_0^\phi U^{-1}|v\rangle = |v\rangle - \left(1 - e^{i\phi}\right) \overline{\langle v|U|v\rangle} U|0\rangle$$

$$|v\rangle = \sum_{i=1}^{N-1} \overline{\langle v|U|i\rangle} U|i\rangle + \overline{\langle v|U|0\rangle} U|0\rangle$$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

$$Q(\phi, \tau) = -US_0^{\phi} U^{-1} S_G^{\tau}$$

S_X^{ϕ} can be implemented efficiently using an ancilla qubit as shown previously

$$S_X^{\phi} = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

Given any quantum state $|v\rangle$, we can write the general form of the relation obtained for $\phi = \pi$

$$US_0^{\phi} U^{-1}|v\rangle = |v\rangle - \left(1 - e^{i\phi}\right) \overline{\langle v|U|v\rangle} U|0\rangle$$

This can be derived by representing $|v\rangle$ in the $U|i\rangle$ basis and applying $US_0^{\phi} U^{-1}$

$$|v\rangle = \sum_{i=1}^{N-1} \overline{\langle v|U|i\rangle} U|i\rangle + \overline{\langle v|U|0\rangle} U|0\rangle$$

$$US_0^{\phi} U^{-1}|v\rangle = US_0^{\phi} \left(\sum_{i=1}^{N-1} \overline{\langle v|U|i\rangle} |i\rangle + \overline{\langle v|U|0\rangle} |0\rangle \right)$$

Modifying the last step

A second method is to modify just the last step to obtain $g_i \equiv 1$

Consider transformations of the form

$$Q(\phi,\tau) = -US_0^{\phi}U^{-1}S_G^{\tau}$$

S_X^{ϕ} can be implemented efficiently using an ancilla qubit as shown previously

$$S_X^{\phi} = \begin{cases} e^{i\phi}|x\rangle & |x\rangle \in X \\ |x\rangle & |x\rangle \notin X \end{cases}$$

Given any quantum state $|v\rangle$, we can write the general form of the relation obtained for $\phi = \pi$

$$US_0^{\phi}U^{-1}|v\rangle = |v\rangle - \left(1 - e^{i\phi}\right)\overline{\langle v|U|v\rangle}U|0\rangle$$

This can be derived by representing $|v\rangle$ in the $U|i\rangle$ basis and applying $US_0^{\phi}U^{-1}$

$$|v\rangle = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + \overline{\langle v|U|0\rangle}U|0\rangle$$

$$US_0^{\phi}U^{-1}|v\rangle = US_0^{\phi}\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}|0\rangle\right) = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|Ue^{i\phi}|0\rangle}|0\rangle\right)$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right)$$

Modifying the last step

$$US_0^\phi U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1 - e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1 - e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Now applying $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ on the superposition $|v\rangle = g|v_G\rangle + b|v_B\rangle$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1 - e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Now applying $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ on the superposition $|v\rangle = g|v_G\rangle + b|v_B\rangle$

$$Q(\phi,\tau)|v\rangle = g\left[-e^{i\tau}|v_G\rangle + e^{i\tau}(1-e^{i\phi})\overline{\langle v_G|U|0\rangle}U|0\rangle\right] + b\left[-|v_B\rangle + (1-e^{i\phi})\overline{\langle v_B|U|0\rangle}U|0\rangle\right]$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1 - e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Now applying $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ on the superposition $|v\rangle = g|v_G\rangle + b|v_B\rangle$

$$Q(\phi,\tau)|v\rangle = g\big[-e^{i\tau}|v_G\rangle + e^{i\tau}(1-e^{i\phi})\overline{\langle v_G|U|0\rangle}U|0\rangle\big] + b\big[-|v_B\rangle + (1-e^{i\phi})\overline{\langle v_B|U|0\rangle}U|0\rangle\big]$$

After $s = \lfloor \frac{\pi}{4\theta} - \frac{1}{2}\rfloor$ iterations of amplification, we have the state

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1 - e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Now applying $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ on the superposition $|v\rangle = g|v_G\rangle + b|v_B\rangle$

$$Q(\phi,\tau)|v\rangle = g\left[-e^{i\tau}|v_G\rangle + e^{i\tau}(1-e^{i\phi})\overline{\langle v_G|U|0\rangle}U|0\rangle\right] + b\left[-|v_B\rangle + (1-e^{i\phi})\overline{\langle v_B|U|0\rangle}U|0\rangle\right]$$

After $s = \lfloor\frac{\pi}{4\theta} - \frac{1}{2}\rfloor$ iterations of amplification, we have the state

$$|\psi_s\rangle = \sin[(2s+1)\theta]|\psi_G\rangle + \cos[(2s+1)\theta]|\psi_B\rangle, \quad \sin\theta = \sqrt{t} = g_0$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1-e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Now applying $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ on the superposition $|v\rangle = g|v_G\rangle + b|v_B\rangle$

$$Q(\phi,\tau)|v\rangle = g\big[-e^{i\tau}|v_G\rangle + e^{i\tau}(1-e^{i\phi})\overline{\langle v_G|U|0\rangle}U|0\rangle\big] + b\big[-|v_B\rangle + (1-e^{i\phi})\overline{\langle v_B|U|0\rangle}U|0\rangle\big]$$

After $s = \lfloor \frac{\pi}{4\theta} - \frac{1}{2}\rfloor$ iterations of amplification, we have the state

$$|\psi_s\rangle = \sin[(2s+1)\theta]|\psi_G\rangle + \cos[(2s+1)\theta]|\psi_B\rangle, \quad \sin\theta = \sqrt{t} = g_0$$

$$Q(\phi,\tau)|\psi_G\rangle = e^{i\tau}\left[(1-e^{i\phi})g_0^2 - 1\right]|\psi_G\rangle + e^{i\tau}(1-e^{i\phi})g_0 b_0|\psi_B\rangle$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1-e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Now applying $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ on the superposition $|v\rangle = g|v_G\rangle + b|v_B\rangle$

$$Q(\phi,\tau)|v\rangle = g\left[-e^{i\tau}|v_G\rangle + e^{i\tau}(1-e^{i\phi})\overline{\langle v_G|U|0\rangle}U|0\rangle\right] + b\left[-|v_B\rangle + (1-e^{i\phi})\overline{\langle v_B|U|0\rangle}U|0\rangle\right]$$

After $s = \lfloor \frac{\pi}{4\theta} - \frac{1}{2}\rfloor$ iterations of amplification, we have the state

$$|\psi_s\rangle = \sin[(2s+1)\theta]|\psi_G\rangle + \cos[(2s+1)\theta]|\psi_B\rangle, \quad \sin\theta = \sqrt{t} = g_0$$

$$Q(\phi,\tau)|\psi_G\rangle = e^{i\tau}\left[(1-e^{i\phi})g_0^2 - 1\right]|\psi_G\rangle + e^{i\tau}(1-e^{i\phi})g_0 b_0|\psi_B\rangle$$

$$Q(\phi,\tau)|\psi_B\rangle = e^{i\tau}(1-e^{i\phi})b_0 g_0|\psi_G\rangle + e^{i\tau}\left[(1-e^{i\phi})b_0^2 - 1\right]|\psi_B\rangle$$

Modifying the last step

$$US_0^{\phi}U^{-1}|v\rangle = U\left(\sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}|i\rangle + \overline{\langle v|U|0\rangle}e^{i\phi}|0\rangle\right) = \sum_{i=1}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle$$

$$= \sum_{i=0}^{N-1}\overline{\langle v|U|i\rangle}U|i\rangle - \overline{\langle v|U|0\rangle}U|0\rangle + e^{i\phi}\overline{\langle v|U|0\rangle}U|0\rangle = |v\rangle - \left(1-e^{i\phi}\right)\overline{\langle v|U|0\rangle}U|0\rangle$$

Now applying $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ on the superposition $|v\rangle = g|v_G\rangle + b|v_B\rangle$

$$Q(\phi,\tau)|v\rangle = g\left[-e^{i\tau}|v_G\rangle + e^{i\tau}(1-e^{i\phi})\overline{\langle v_G|U|0\rangle}U|0\rangle\right] + b\left[-|v_B\rangle + (1-e^{i\phi})\overline{\langle v_B|U|0\rangle}U|0\rangle\right]$$

After $s = \lfloor \frac{\pi}{4\theta} - \frac{1}{2} \rfloor$ iterations of amplification, we have the state

$$|\psi_s\rangle = \sin[(2s+1)\theta]|\psi_G\rangle + \cos[(2s+1)\theta]|\psi_B\rangle, \quad \sin\theta = \sqrt{t} = g_0$$

$$Q(\phi,\tau)|\psi_G\rangle = e^{i\tau}\left[(1-e^{i\phi})g_0^2 - 1\right]|\psi_G\rangle + e^{i\tau}(1-e^{i\phi})g_0b_0|\psi_B\rangle$$

$$Q(\phi,\tau)|\psi_B\rangle = e^{i\tau}(1-e^{i\phi})b_0g_0|\psi_G\rangle + e^{i\tau}\left[(1-e^{i\phi})b_0^2 - 1\right]|\psi_B\rangle$$

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

$$g(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left[\left(1-e^{i\phi}\right)g_0^2 - 1\right] + \cos[(2s+1)\theta]\left(1-e^{i\phi}\right)b_0 g_0$$

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

$$g(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left[\left(1-e^{i\phi}\right)g_0^2 - 1\right] + \cos[(2s+1)\theta]\left(1-e^{i\phi}\right)b_0 g_0$$
$$b(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right]$$

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

$$g(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left[\left(1-e^{i\phi}\right)g_0^2 - 1\right] + \cos[(2s+1)\theta]\left(1-e^{i\phi}\right)b_0 g_0$$
$$b(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right]$$

The goal is to find values for ϕ and τ such that when the final iteration $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ is applied the solution is obtained with certainty

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

$$g(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left[\left(1-e^{i\phi}\right)g_0^2-1\right] + \cos[(2s+1)\theta]\left(1-e^{i\phi}\right)b_0 g_0$$
$$b(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2-1\right]$$

The goal is to find values for ϕ and τ such that when the final iteration $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ is applied the solution is obtained with certainty

This boils down to finding a solution to $b(\phi,\tau)=0$ and recalling that $b_0=\sqrt{1-g_0^2}$

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

$$g(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left[\left(1-e^{i\phi}\right)g_0^2 - 1\right] + \cos[(2s+1)\theta]\left(1-e^{i\phi}\right)b_0 g_0$$
$$b(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right]$$

The goal is to find values for ϕ and τ such that when the final iteration $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ is applied the solution is obtained with certainty

This boils down to finding a solution to $b(\phi,\tau) = 0$ and recalling that $b_0 = \sqrt{1-g_0^2}$

$$\sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right] = 0$$

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

$$g(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left[\left(1-e^{i\phi}\right)g_0^2 - 1\right] + \cos[(2s+1)\theta]\left(1-e^{i\phi}\right)b_0 g_0$$
$$b(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right]$$

The goal is to find values for ϕ and τ such that when the final iteration $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ is applied the solution is obtained with certainty

This boils down to finding a solution to $b(\phi,\tau) = 0$ and recalling that $b_0 = \sqrt{1-g_0^2}$

$$\sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right] = 0$$
$$e^{i\tau}\left(1-e^{i\phi}\right)g_0\sqrt{1-g_0^2}\,\sin[(2s+1)\theta] = \left[1-\left(1-e^{i\phi}\right)(1-g_0^2)\right]\cos[(2s+1)\theta]$$

Modifying the last step

$$Q(\phi,\tau)|\psi\rangle = g(\phi,\tau)|\psi_G\rangle + b(\phi,\tau)|\psi_B\rangle$$

$$g(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left[\left(1-e^{i\phi}\right)g_0^2 - 1\right] + \cos[(2s+1)\theta]\left(1-e^{i\phi}\right)b_0 g_0$$
$$b(\phi,\tau) = \sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right]$$

The goal is to find values for ϕ and τ such that when the final iteration $Q(\phi,\tau) = US_0^{\phi}U^{-1}S_G^{\tau}$ is applied the solution is obtained with certainty

This boils down to finding a solution to $b(\phi,\tau) = 0$ and recalling that $b_0 = \sqrt{1-g_0^2}$

$$\sin[(2s+1)\theta]e^{i\tau}\left(1-e^{i\phi}\right)g_0 b_0 + \cos[(2s+1)\theta]\left[\left(1-e^{i\phi}\right)b_0^2 - 1\right] = 0$$
$$e^{i\tau}\left(1-e^{i\phi}\right)g_0\sqrt{1-g_0^2}\,\sin[(2s+1)\theta] = \left[1-\left(1-e^{i\phi}\right)\left(1-g_0^2\right)\right]\cos[(2s+1)\theta]$$
$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi}) + e^{i\phi}}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$
which then gives

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1} = \frac{4g_0^2 b_0^2}{(2g_0^2-1)^2}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1} = \frac{4g_0^2 b_0^2}{(2g_0^2-1)^2}$$

Taking the square root gives the maximum magnitude

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1} = \frac{4g_0^2 b_0^2}{(2g_0^2-1)^2}$$

Taking the square root gives the maximum magnitude

$$\frac{2g_0 b_0}{2g_0^2-1}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1} = \frac{4g_0^2 b_0^2}{(2g_0^2-1)^2}$$

Taking the square root gives the maximum magnitude

$$\frac{2g_0 b_0}{2g_0^2-1} = \frac{2g_0 b_0}{g_0^2-b_0^2}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1} = \frac{4g_0^2 b_0^2}{(2g_0^2-1)^2}$$

Taking the square root gives the maximum magnitude and recalling that $g_0 = \sin\theta$

$$\frac{2g_0 b_0}{2g_0^2-1} = \frac{2g_0 b_0}{g_0^2-b_0^2}$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1} = \frac{4g_0^2 b_0^2}{(2g_0^2-1)^2}$$

Taking the square root gives the maximum magnitude and recalling that $g_0 = \sin\theta$

$$\frac{2g_0 b_0}{2g_0^2-1} = \frac{2g_0 b_0}{g_0^2-b_0^2} = \tan(2\theta)$$

Modifying the last step

$$\cot[(2s+1)\theta] = \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

The right side can be any value from 0 to $\tan(2\theta)$ so the last step can be adjusted as needed

First find ϕ and then choose τ to make the right side of the equation real

Compute the modulus squared of the right side of the equation to make it real

$$\frac{e^{-i\tau}(1-e^{-i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{-i\phi})+e^{-i\phi}} \cdot \frac{e^{i\tau}(1-e^{i\phi})g_0\sqrt{1-g_0^2}}{g_0^2(1-e^{i\phi})+e^{i\phi}}$$

$$= \frac{(2-e^{i\phi}-e^{-i\phi})g_0^2(1-g_0^2)}{g_0^4(2-e^{i\phi}-e^{-i\phi})+g_0^2(e^{i\phi}+e^{-i\phi}-2)+1} = \frac{g_0^2 b_0^2(2-2\cos\phi)}{g_0^4(2-2\cos\phi)-g_0^2(2-2\cos\phi)+1}$$

This can be maximized when $\cos\phi = -1$ which then gives

$$\frac{4g_0^2 b_0^2}{4g_0^4-4g_0^2+1} = \frac{4g_0^2 b_0^2}{(2g_0^2-1)^2}$$

Taking the square root gives the maximum magnitude and recalling that $g_0 = \sin\theta$

$$\frac{2g_0 b_0}{2g_0^2-1} = \frac{2g_0 b_0}{g_0^2-b_0^2} = \tan(2\theta)$$

Unknown number of solutions

In order to achieve a solution with certainty using a Grover-like search algorithm, it is necessary to know how many iterations to apply

Unknown number of solutions

In order to achieve a solution with certainty using a Grover-like search algorithm, it is necessary to know how many iterations to apply

This requires a knowledge of g_0, which is a measure of how many solutions exist, so what if the number of solutions is unknown?

Unknown number of solutions

In order to achieve a solution with certainty using a Grover-like search algorithm, it is necessary to know how many iterations to apply

This requires a knowledge of g_0, which is a measure of how many solutions exist, so what if the number of solutions is unknown?

Two methods are available: (1) repeated searches with a random number of iterations of Q, and (2) quantum counting using the quantum Fourier transform

Unknown number of solutions

In order to achieve a solution with certainty using a Grover-like search algorithm, it is necessary to know how many iterations to apply

This requires a knowledge of g_0, which is a measure of how many solutions exist, so what if the number of solutions is unknown?

Two methods are available: (1) repeated searches with a random number of iterations of Q, and (2) quantum counting using the quantum Fourier transform

Consider a problem with tN solutions and t is unknown

Unknown number of solutions

In order to achieve a solution with certainty using a Grover-like search algorithm, it is necessary to know how many iterations to apply

This requires a knowledge of g_0, which is a measure of how many solutions exist, so what if the number of solutions is unknown?

Two methods are available: (1) repeated searches with a random number of iterations of Q, and (2) quantum counting using the quantum Fourier transform

Consider a problem with tN solutions and t is unknown

The strategy is to repeatedly execute Grover's algorithm with a number of iterations chosen randomly from between 0 and $\frac{\pi}{4}\sqrt{N}$ times

Unknown number of solutions

In order to achieve a solution with certainty using a Grover-like search algorithm, it is necessary to know how many iterations to apply

This requires a knowledge of g_0, which is a measure of how many solutions exist, so what if the number of solutions is unknown?

Two methods are available: (1) repeated searches with a random number of iterations of Q, and (2) quantum counting using the quantum Fourier transform

Consider a problem with tN solutions and t is unknown

The strategy is to repeatedly execute Grover's algorithm with a number of iterations chosen randomly from between 0 and $\frac{\pi}{4}\sqrt{N}$ times

The average probability of success for a run with a randomly chosen i iterations of Q between 0 and r is

Unknown number of solutions

In order to achieve a solution with certainty using a Grover-like search algorithm, it is necessary to know how many iterations to apply

This requires a knowledge of g_0, which is a measure of how many solutions exist, so what if the number of solutions is unknown?

Two methods are available: (1) repeated searches with a random number of iterations of Q, and (2) quantum counting using the quantum Fourier transform

Consider a problem with tN solutions and t is unknown

The strategy is to repeatedly execute Grover's algorithm with a number of iterations chosen randomly from between 0 and $\frac{\pi}{4}\sqrt{N}$ times

The average probability of success for a run with a randomly chosen i iterations of Q between 0 and r is

$$P_r(i < r) = \frac{1}{r}\sum_{i=0}^{r-1} \sin^2[(2i+1)\theta], \quad \sin\theta = \sqrt{t}$$

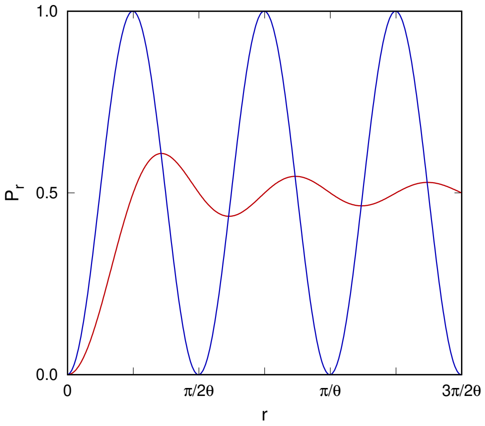
Repeated random searches

$$P_r(i < r) = \frac{1}{r}\sum_{i=0}^{r-1} \sin^2[(2i+1)\theta]$$

Repeated random searches

$$P_r(i < r) = \frac{1}{r}\sum_{i=0}^{r-1} \sin^2[(2i+1)\theta]$$

For example, for $t = 0.0001$ and $\theta \approx 0.01$ we can plot the probability P_r of finding the solution when choosing a random number of iterations along choosing exactly r iterations as a function of r

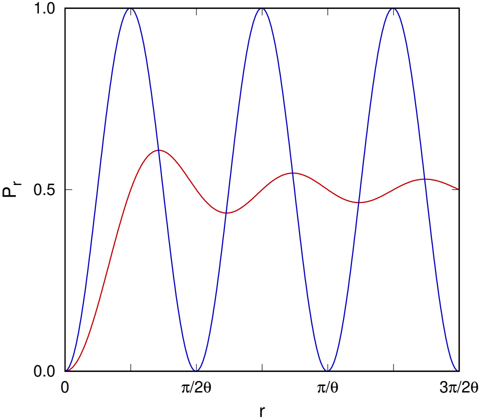


Repeated random searches

$$P_r(i < r) = \frac{1}{r}\sum_{i=0}^{r-1} \sin^2[(2i+1)\theta]$$

For example, for $t = 0.0001$ and $\theta \approx 0.01$ we can plot the probability P_r of finding the solution when choosing a random number of iterations along choosing exactly r iterations as a function of r

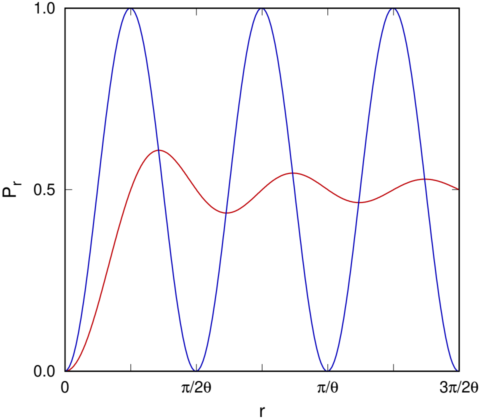
For all $r \geq \frac{\pi}{4}\sqrt{\frac{1}{t}} \approx \frac{\pi}{4\theta}$ there is a constant c such that $P_r(i < r) > c$



Repeated random searches

$$P_r(i < r) = \frac{1}{r}\sum_{i=0}^{r-1} \sin^2[(2i+1)\theta]$$

For example, for $t = 0.0001$ and $\theta \approx 0.01$ we can plot the probability P_r of finding the solution when choosing a random number of iterations along choosing exactly r iterations as a function of r



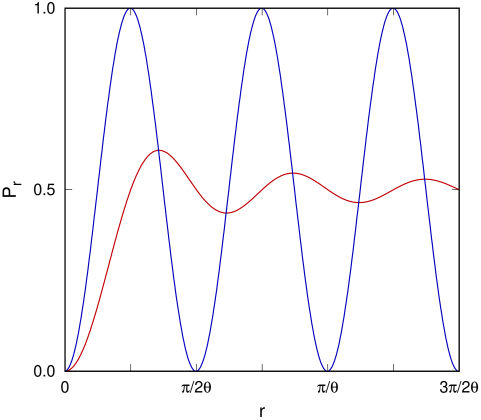
For all $r \geq \frac{\pi}{4}\sqrt{\frac{1}{t}} \approx \frac{\pi}{4\theta}$ there is a constant c such that $P_r(i < r) > c$

If $\frac{1}{t} \leq N$, that is if there is at least one solution, choosing $r = \frac{\pi}{4}\sqrt{N}$ guarantees a probability of at least c of finding the solution with a single run of the algorithm

Repeated random searches

$$P_r(i < r) = \frac{1}{r}\sum_{i=0}^{r-1} \sin^2[(2i+1)\theta]$$

For example, for $t = 0.0001$ and $\theta \approx 0.01$ we can plot the probability P_r of finding the solution when choosing a random number of iterations along choosing exactly r iterations as a function of r



For all $r \geq \frac{\pi}{4}\sqrt{\frac{1}{t}} \approx \frac{\pi}{4\theta}$ there is a constant c such that $P_r(i < r) > c$

If $\frac{1}{t} \leq N$, that is if there is at least one solution, choosing $r = \frac{\pi}{4}\sqrt{N}$ guarantees a probability of at least c of finding the solution with a single run of the algorithm

This means that the expected number of calls to the oracle is $O(\sqrt{N})$

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

If U and Q be defined as for Grover's algorithm,
define a transformation R_Q with inputs $|k\rangle$ and $|\psi\rangle$

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

If U and Q be defined as for Grover's algorithm, define a transformation R_Q with inputs $|k\rangle$ and $|\psi\rangle$

$$R_Q : |k\rangle \otimes |\psi\rangle \longrightarrow |k\rangle \otimes Q^k|\psi\rangle$$

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

If U and Q be defined as for Grover's algorithm, define a transformation R_Q with inputs $|k\rangle$ and $|\psi\rangle$

$$R_Q : |k\rangle \otimes |\psi\rangle \longrightarrow |k\rangle \otimes Q^k|\psi\rangle$$

Apply R_Q to a superposition of all $k < M = 2^m$ tensored with $U|0\rangle$

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

If U and Q be defined as for Grover's algorithm, define a transformation R_Q with inputs $|k\rangle$ and $|\psi\rangle$

$$R_Q : |k\rangle \otimes |\psi\rangle \quad \longrightarrow \quad |k\rangle \otimes Q^k|\psi\rangle$$

Apply R_Q to a superposition of all $k < M = 2^m$ tensored with $U|0\rangle$

$$\frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} |k\rangle \otimes U|0\rangle \quad \xrightarrow{R_Q} \quad \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} |k\rangle \otimes \big(g_k|\psi_G\rangle + b_k|\psi_B\rangle\big)$$

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

If U and Q be defined as for Grover's algorithm, define a transformation R_Q with inputs $|k\rangle$ and $|\psi\rangle$

$$R_Q : |k\rangle \otimes |\psi\rangle \quad \longrightarrow \quad |k\rangle \otimes Q^k|\psi\rangle$$

Apply R_Q to a superposition of all $k < M = 2^m$ tensored with $U|0\rangle$

$$\frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} |k\rangle \otimes U|0\rangle \quad \xrightarrow{R_Q} \quad \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} |k\rangle \otimes \left(g_k|\psi_G\rangle + b_k|\psi_B\rangle\right)$$

Measureing the right register gives a state $|x\rangle$ that is a good state or a bad state so that the left register collapses to

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

If U and Q be defined as for Grover's algorithm, define a transformation R_Q with inputs $|k\rangle$ and $|\psi\rangle$

$$R_Q : |k\rangle \otimes |\psi\rangle \quad \longrightarrow \quad |k\rangle \otimes Q^k|\psi\rangle$$

Apply R_Q to a superposition of all $k < M = 2^m$ tensored with $U|0\rangle$

$$\frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}|k\rangle \otimes U|0\rangle \quad \xrightarrow{R_Q} \quad \frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}|k\rangle \otimes \big(g_k|\psi_G\rangle + b_k|\psi_B\rangle\big)$$

Measureing the right register gives a state $|x\rangle$ that is a good state or a bad state so that the left register collapses to

$$|\psi\rangle = C\sum_{k=0}^{M-1} b_k|k\rangle$$

Quantum counting

Instead of brute force repetition, quantum counting is a more quantum-like approach which is also $O(\sqrt{N})$ in oracle queries

The strategy is to create a superposition of results for different applications of Q and then use the quantum Fourier transform to get an estimate of the relative number of solutions t which can then be used to apply Grover's algorithm optimally

If U and Q be defined as for Grover's algorithm, define a transformation R_Q with inputs $|k\rangle$ and $|\psi\rangle$

$$R_Q : |k\rangle \otimes |\psi\rangle \quad \longrightarrow \quad |k\rangle \otimes Q^k|\psi\rangle$$

Apply R_Q to a superposition of all $k < M = 2^m$ tensored with $U|0\rangle$

$$\frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} |k\rangle \otimes U|0\rangle \quad \xrightarrow{R_Q} \quad \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} |k\rangle \otimes \big(g_k|\psi_G\rangle + b_k|\psi_B\rangle\big)$$

Measureing the right register gives a state $|x\rangle$ that is a good state or a bad state so that the left register collapses to

$$|\psi\rangle = C \sum_{k=0}^{M-1} b_k|k\rangle \quad \text{or} \quad |\psi\rangle' = C' \sum_{k=0}^{M-1} g_k|k\rangle$$

Quantum counting

$$|\psi\rangle = C \sum_{k=0}^{M-1} b_k |k\rangle \quad , \quad |\psi\rangle' = C' \sum_{k=0}^{M-1} g_k |k\rangle$$

Quantum counting

$$|\psi\rangle = C \sum_{k=0}^{M-1} b_k |k\rangle \quad , \quad |\psi\rangle' = C' \sum_{k=0}^{M-1} g_k |k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Quantum counting

$$|\psi\rangle = C \sum_{k=0}^{M-1} b_k |k\rangle = C \sum_{k=0}^{M-1} \cos[(2k+1)\theta] |k\rangle, \quad |\psi\rangle' = C' \sum_{k=0}^{M-1} g_k |k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Quantum counting

$$|\psi\rangle = C \sum_{k=0}^{M-1} b_k |k\rangle = C \sum_{k=0}^{M-1} \cos[(2k+1)\theta] |k\rangle, \quad |\psi\rangle' = C' \sum_{k=0}^{M-1} g_k |k\rangle = C' \sum_{k=0}^{M-1} \sin[(2k+1)\theta] |k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Quantum counting

$$|\psi\rangle = C\sum_{k=0}^{M-1} b_k|k\rangle = C\sum_{k=0}^{M-1}\cos[(2k+1)\theta]|k\rangle, \quad |\psi\rangle' = C'\sum_{k=0}^{M-1} g_k|k\rangle = C'\sum_{k=0}^{M-1}\sin[(2k+1)\theta]|k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Suppose $|\psi\rangle$ is the result, we apply the quantum Fourier transform to get

Quantum counting

$$|\psi\rangle = C\sum_{k=0}^{M-1} b_k|k\rangle = C\sum_{k=0}^{M-1}\cos[(2k+1)\theta]|k\rangle, \quad |\psi\rangle' = C'\sum_{k=0}^{M-1} g_k|k\rangle = C'\sum_{k=0}^{M-1}\sin[(2k+1)\theta]|k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Suppose $|\psi\rangle$ is the result, we apply the quantum Fourier transform to get

$$\mathcal{F} : C\sum_{k=0}^{M-1} b_k|k\rangle \longrightarrow \sum_{j=0}^{M-1} B_j|j\rangle$$

Quantum counting

$$|\psi\rangle = C\sum_{k=0}^{M-1} b_k|k\rangle = C\sum_{k=0}^{M-1}\cos[(2k+1)\theta]|k\rangle, \quad |\psi\rangle' = C'\sum_{k=0}^{M-1} g_k|k\rangle = C'\sum_{k=0}^{M-1}\sin[(2k+1)\theta]|k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Suppose $|\psi\rangle$ is the result, we apply the quantum Fourier transform to get

$$\mathcal{F} : C\sum_{k=0}^{M-1} b_k|k\rangle \longrightarrow \sum_{j=0}^{M-1} B_j|j\rangle$$

Recall from the previous discussion that for a cosine function of period $\frac{\pi}{\theta}$, most of the amplitude appears in the B_j which are close to the single value $\frac{M\theta}{\pi}$

Quantum counting

$$|\psi\rangle = C\sum_{k=0}^{M-1} b_k|k\rangle = C\sum_{k=0}^{M-1}\cos[(2k+1)\theta]|k\rangle, \quad |\psi\rangle' = C'\sum_{k=0}^{M-1} g_k|k\rangle = C'\sum_{k=0}^{M-1}\sin[(2k+1)\theta]|k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Suppose $|\psi\rangle$ is the result, we apply the quantum Fourier transform to get

$$\mathcal{F} : C\sum_{k=0}^{M-1} b_k|k\rangle \longrightarrow \sum_{j=0}^{M-1} B_j|j\rangle$$

Recall from the previous discussion that for a cosine function of period $\frac{\pi}{\theta}$, most of the amplitude appears in the B_j which are close to the single value $\frac{M\theta}{\pi}$

By measuring the Fourier transformed state, we obtain a state $|j\rangle$ which permits good approximation to θ by taking $\theta = \frac{\pi j}{M}$ and thus $t \approx \sqrt{\sin\theta}$ with high probability

Quantum counting

$$|\psi\rangle = C\sum_{k=0}^{M-1} b_k|k\rangle = C\sum_{k=0}^{M-1}\cos[(2k+1)\theta]|k\rangle, \quad |\psi\rangle' = C'\sum_{k=0}^{M-1} g_k|k\rangle = C'\sum_{k=0}^{M-1}\sin[(2k+1)\theta]|k\rangle$$

We know that $g_k = \sin[(2k+1)\theta]$ and $b_k = \cos[(2k+1)\theta]$ so we can write

Suppose $|\psi\rangle$ is the result, we apply the quantum Fourier transform to get

$$\mathcal{F} : C\sum_{k=0}^{M-1} b_k|k\rangle \longrightarrow \sum_{j=0}^{M-1} B_j|j\rangle$$

Recall from the previous discussion that for a cosine function of period $\frac{\pi}{\theta}$, most of the amplitude appears in the B_j which are close to the single value $\frac{M\theta}{\pi}$

By measuring the Fourier transformed state, we obtain a state $|j\rangle$ which permits good approximation to θ by taking $\theta = \frac{\pi j}{M}$ and thus $t \approx \sqrt{\sin\theta}$ with high probability

M can be determined by repeating this algorithm with increasing values of M until j is measured to be a non-zero value