Today's outline - March 10, 2022

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- Introduction to Grover's algorithm

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Homework Assignment #06:
TBA on (still working on it)

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Quantum circuit simulator https://algassert.com/quirk

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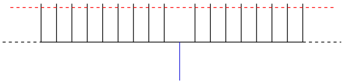

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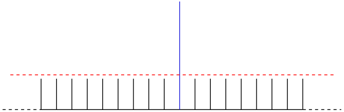

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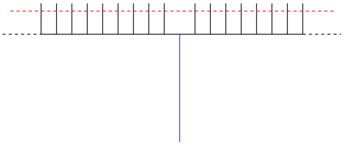

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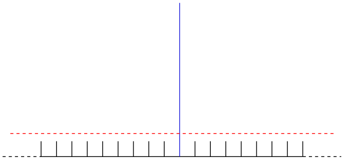

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Grover's algorithm is achieved by repeated application of a unitary transformation, $Q : g_i|\psi_G\rangle + b_i|\psi_B\rangle \to g_{+1}|\psi_G\rangle + b_{i+1}|\psi_B\rangle$ such that the g_i increase to a maximal value with corresponding decrease in the b_i

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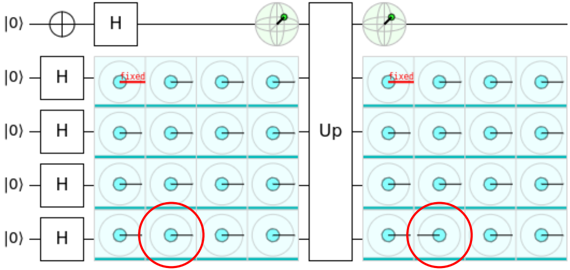
https://tinyurl.com/yc32wavm



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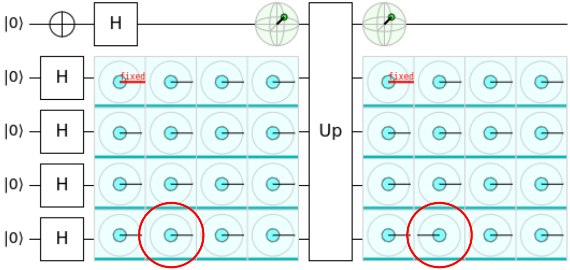
Only one of the superimposed states has its sign flipped and the ancilla bit is unchanged



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Suppose there are J negative and K positive coefficients each with magnitude $\frac{1}{\sqrt{N}}$

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Since $J \ll K$, the c_j grow and the c_k shrink quickly with each iteration

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$$Q = -WS_0^\pi WS_G^\pi$$

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The iterative operator $Q = DS_G^{\pi}$ transforms $g_i|\psi_G\rangle + b_i|\psi_B\rangle$ to $g_{i+1}|\psi_G\rangle + b_{i+1}|\psi_B\rangle$ in two steps, first

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These relations permit the calculation of the optimal number of iterations

Computing the number of iterations

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https://tinyurl.com/5n7jcpuh