Today's outline - February 22, 2022

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- A word about hands-on component

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Homework Assignment #05:
Chapter 7:1,3,4
due Thursday, March 03, 2022

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Quantum circuit simulator https://algassert.com/quirk

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$$a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1} \longrightarrow |\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i, i\rangle$$

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$$\langle x, x|\psi\rangle = 0$$

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All are zero except for when $x - u = 0 \mod N$ so the only term which survives is $A(u)$

DFT example

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The fast Fourier transform (FFT) is an efficient implementation for $N = 2^n$, called $F^{(n)}$

The implementation involves recursive decomposition of $F^{(n)}$ in terms of Fourier transforms of lower powers of 2

If $\omega_{(n)} = e^{2\pi i/N}$ is the Nth root of unity, the elements of the Fourier transform matrix are

$$F^{(n)}_{xy} = \omega^{xy}_{(n)}$$

where $x, y \in \{0, \ldots, N-1\}$

Let $F^{(k)}$, $I^{(k)}$, $R^{(k)}$ be the 2^k-dimensional Fourier transform matrix, identity matrix, and a permutation matrix defined by

$$R^{(k)}_{xy} = \begin{cases} 1 & \text{for } 2x = y \\ 1 & \text{for } 2x - 2^k + 1 = y \\ 0 & \text{otherwise} \end{cases}$$

$$x = 2; \quad y = 2x = 4$$

$$R^{(3)} = \begin{pmatrix} 1 & & & & & & & \\ & & 1 & & & & & \\ & & & & 1 & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$

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$x = 3; \quad y = 2x = 6$

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$$x = 4; \quad y = 2x - 8 + 1 = 1$$

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$$x = 5; \quad y = 2x - 8 + 1 = 3$$

$$R^{(3)} = \begin{pmatrix} 1 & & & & & & & \\ & & 1 & & & & & \\ & & & & 1 & & & \\ & & & & & & 1 & \\ & 1 & & & & & & \\ & & & 1 & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$

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The $R^{(k)}$ matrix performs a shuffle transform on a column vector

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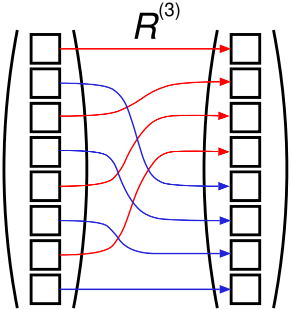

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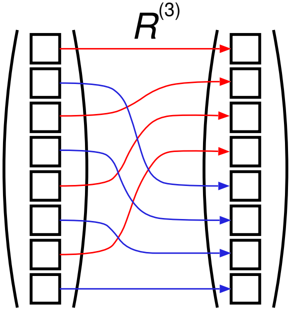
Using $F^{(k)}$, $I^{(k)}$, $R^{(k)}$ plus $D^{(k)}$, a diagonal matrix with entries $\omega^0_{(k+1)}, \ldots, \omega^{2^k-1}_{(k+1)}$ it is possible to solve for $F^{(k)}$ recursively



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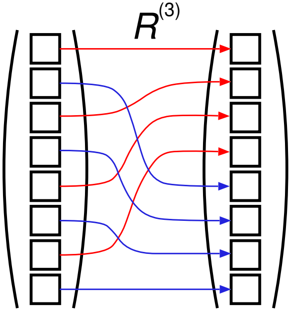

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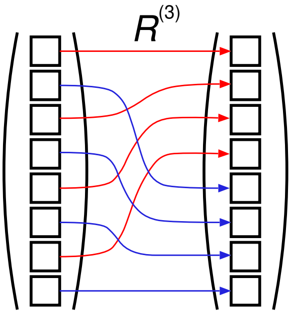

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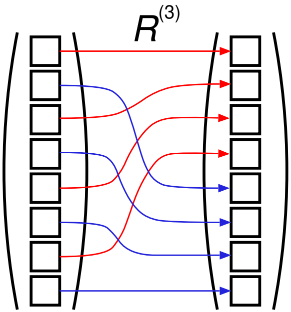

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Computing $F^{(k)}$ becomes computing 2 $F^{(k-1)}$, then 4 $F^{(k-2)}$, and so on until 2^{k-1} $F^{(1)}$ matrices and $O(nN)$ computation

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The QFT is exponentially faster, $[O(n^2)]$, than the discrete $[O(N^2)]$, and the fast $[O(N \log N)]$ transforms

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$$U_F^{(k+1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} I^{(k)} & D^{(k)} \\ I^{(k)} & -D^{(k)} \end{pmatrix} \begin{pmatrix} U_F^{(k)} & 0 \\ 0 & U_F^{(k)} \end{pmatrix} R^{(k+1)}$$

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All matrices are unitary and can be implemented efficiently on a quantum computer

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This can be accomplished by a permutation of the $k+1$ qubits,

$$\begin{aligned} 000 &\rightarrow 000 \\ 001 &\rightarrow 010 \\ 010 &\rightarrow 100 \\ 011 &\rightarrow 110 \\ 100 &\rightarrow 001 \\ 101 &\rightarrow 011 \end{aligned}$$

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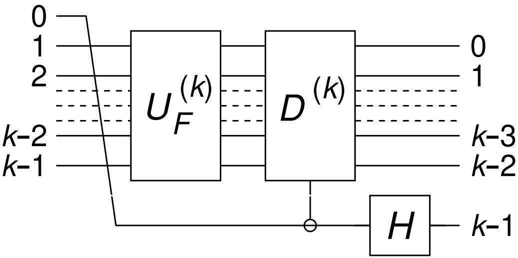

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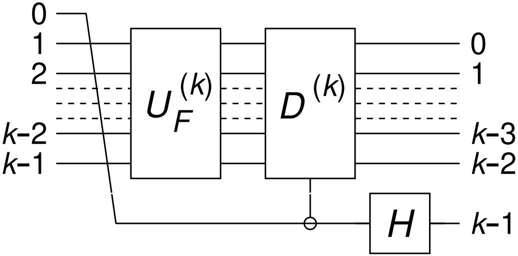

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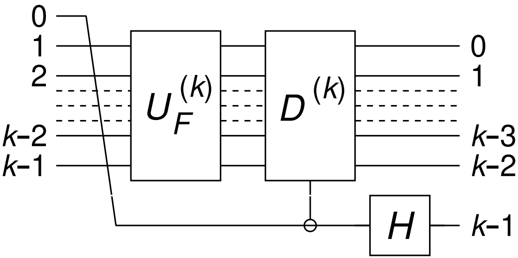

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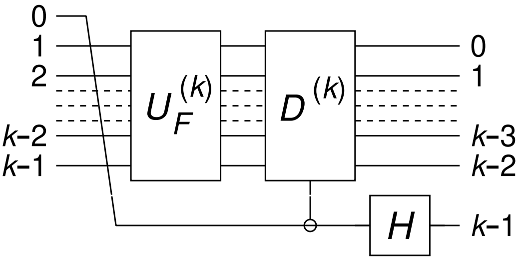

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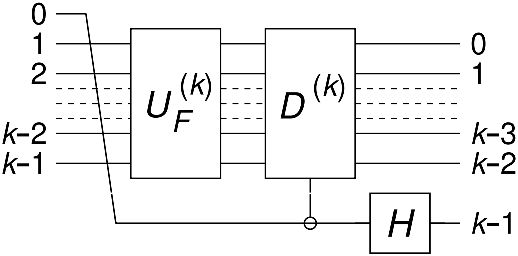

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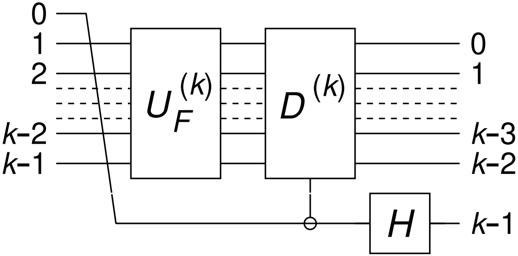

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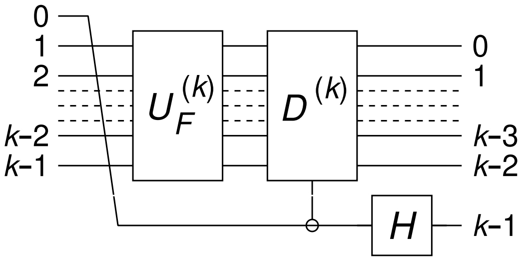

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4. $H|x_{n-1}\rangle$

$D^{(k)}$ and $R^{(k)}$ can be implemented with $O(k)$ gates and the k^{th} step in the recursion adds $O(K)$ gates to the implementation of $U_F^{(n)}$, overall $U_F^{(n)}$ takes $O(n^2)$ gates to implement which exponentially faster than the $O(n2^n)$ for a classical FFT

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A more straightforward circuit provides insight to the recursive one just described

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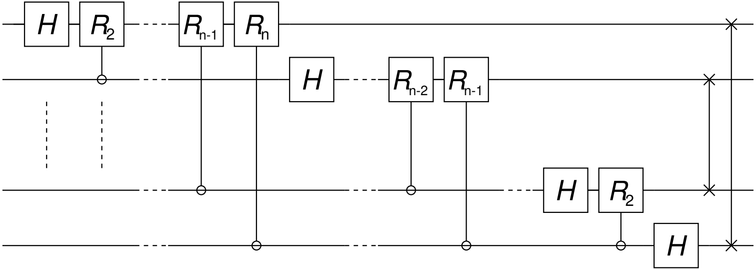

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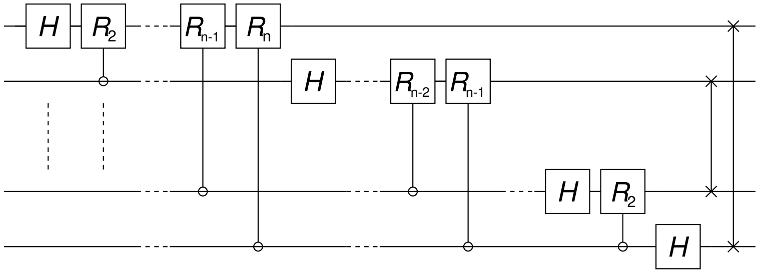

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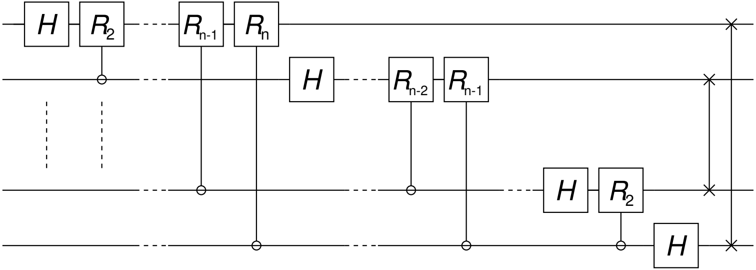

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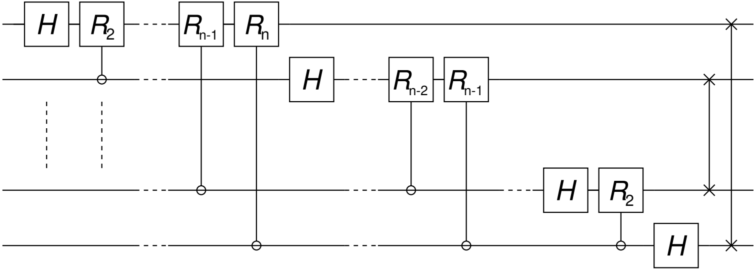

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At the end, all the qubits need to be swapped to recover the proper order