# Today's outline - February 15, 2022

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Reading Assignment: Chapter 7.5-7.7

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Homework Assignment #05:
Chapter 7:1,3,4
due Thursday, February 24, 2022

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For $f(x)$ balanced, only one term picks up a phase shift, giving a result of $|-\rangle|-\rangle$

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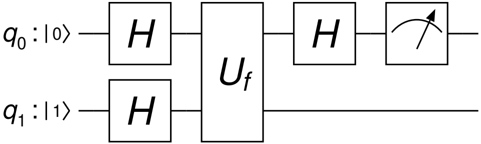
As a quantum circuit, Deutch's algorithm is implemented by



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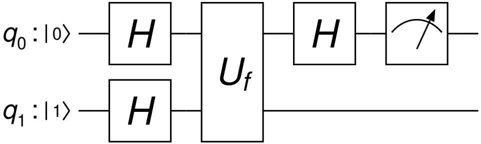

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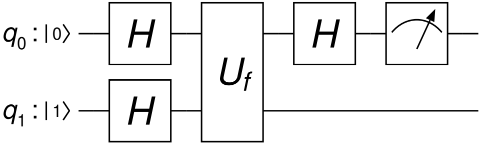

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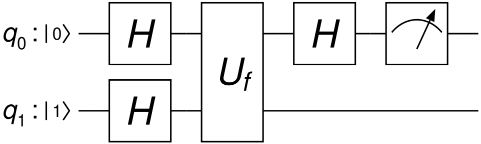

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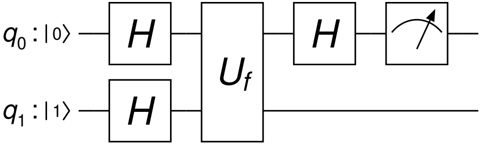

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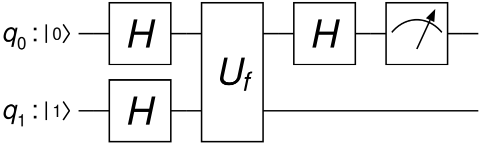

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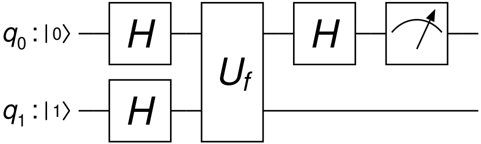


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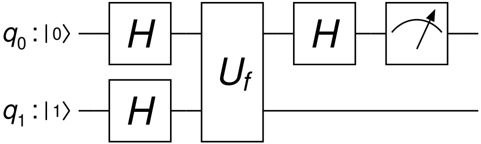


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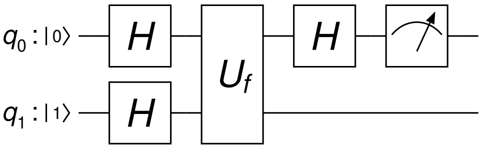


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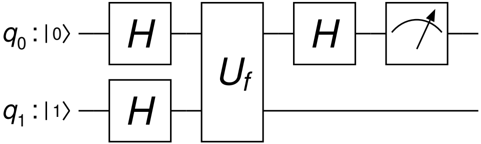


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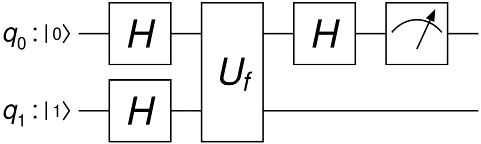


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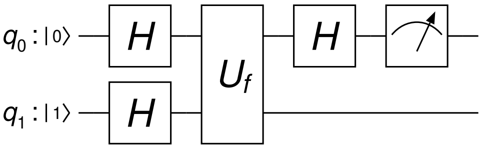


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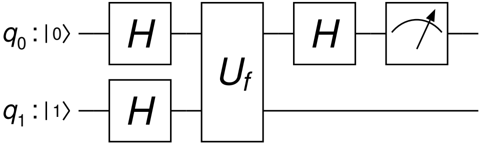


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As a quantum circuit, Deutch's algorithm is implemented by

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4. apply the Hadamard transform to $q_0$
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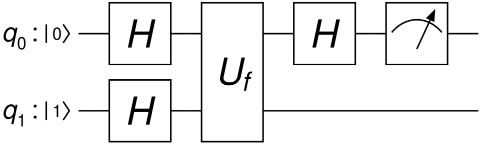


(a) $q_0 = 0 \longrightarrow f(x)$ is constant

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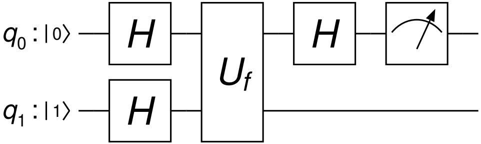


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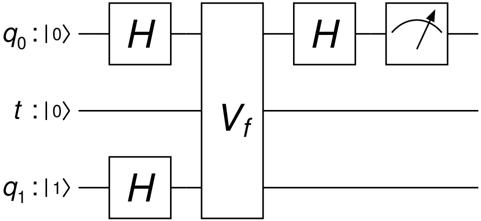

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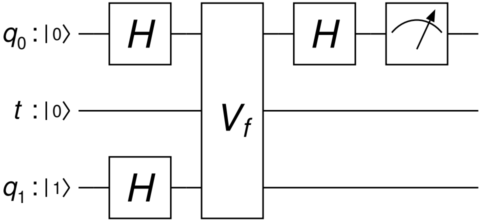

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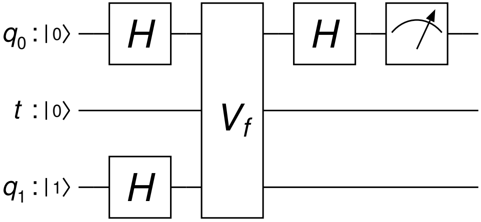

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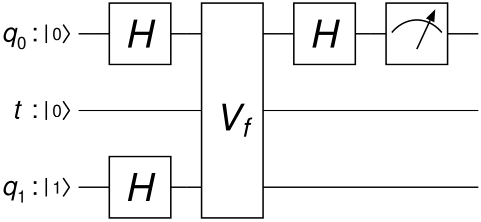


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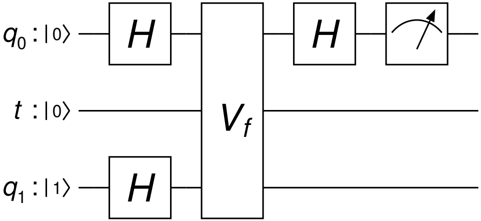


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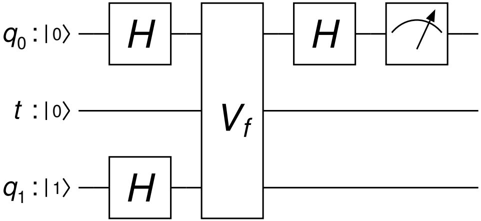


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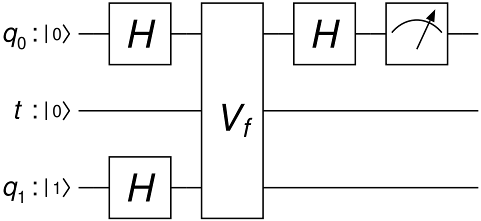


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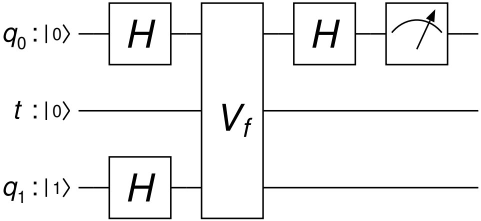


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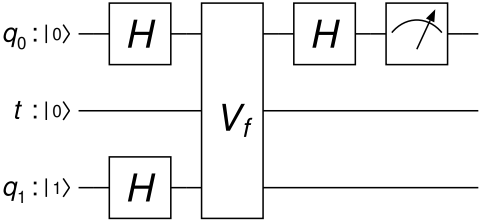


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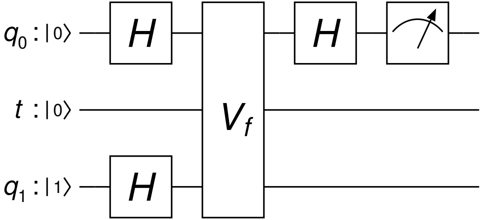

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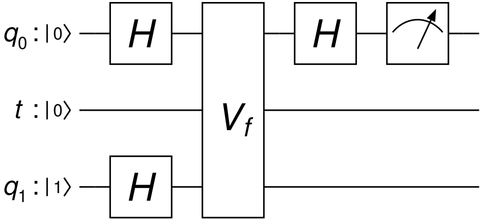

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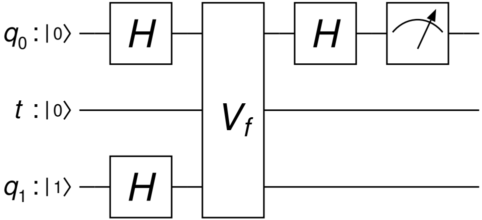


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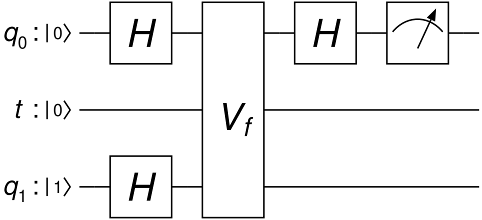


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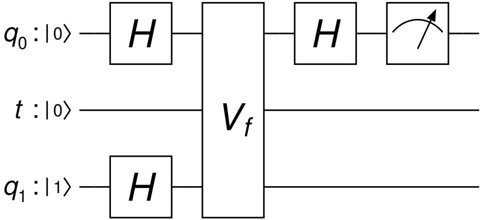


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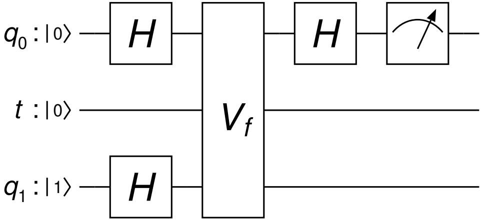


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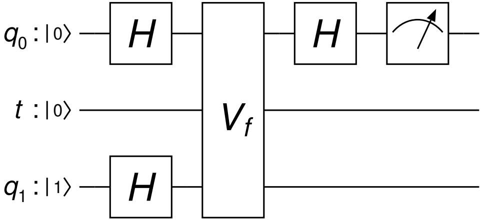


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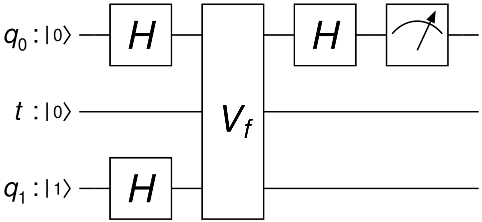


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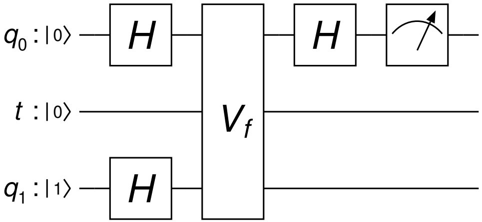


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# Entanglement problems

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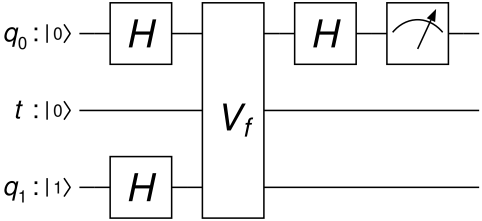


The final step of applying the Hadamard transformation to $q_0$ uses the $H \otimes I \otimes I$ transformation which yields

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The depends on being able to compute membership in $X$ efficiently but if this is possible with a quantum transformation $U_f$ then through the use of a temporary qubit, it is possible to compute $S_X^\phi$

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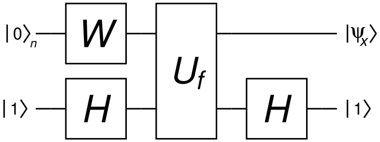

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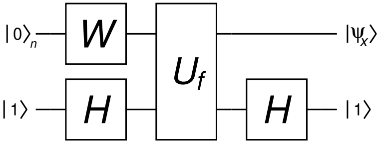


The circuit starts with a uniform superposition of an $n$-qubit register and an acilla qubit in the $|1\rangle$ state to create the superposition $|\psi_X\rangle = \sum(-1)^{f(x)}|x\rangle$

# State-dependent phase changes

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Suppose that $f(x) = x$, we want a subroutine that changes the phase of an s-qubit standard basis state $|x\rangle$ by $\phi(x) = 2\pi x/2^s$ using the transformation

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Using the subroutine $Phase\ : |a\rangle \to e^{i2\pi s/2^s}$ it is now possible to write a program that implements the $n$-qubit transformation $Phase_f\ : |x\rangle \to e^{i2\pi f(x)/2^s}$

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# State-dependent amplitude shifts

We wish to rotate each term in a superposition by a single qubit rotation $R(\beta(x))$ where $\beta(x)$ is state-dependent such that $|x\rangle \otimes |b\rangle \to |x\rangle \otimes (R(\beta(x))|b\rangle$

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