Today's outline - February 03, 2022

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- Multiply controlled operators

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Reading Assignment: Chapter 7.1-7.2

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Reading Assignment: Chapter 7.1-7.2

Homework Assignment #04:
Chapter 5:4,6,9,15,16,17
due Tuesday, February 15, 2022

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- Multiply controlled operators
- Arbitrary controlled operators
- Implementing general operators
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Reading Assignment: Chapter 7.1-7.2

Homework Assignment #04:
Chapter 5:4,6,9,15,16,17
due Tuesday, February 15, 2022

No class on Tuesday, Feb 08, 2022

Multiply controlled transformations

Controlled operations can be generalized to more than one control bit

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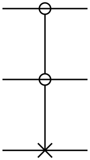

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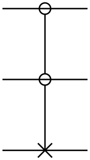

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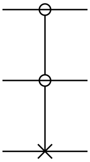

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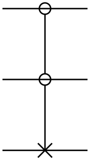

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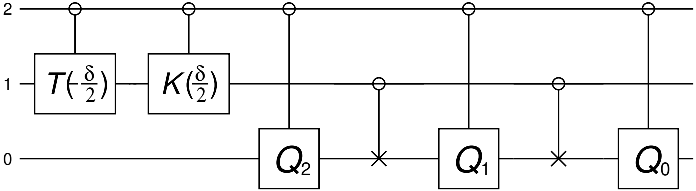

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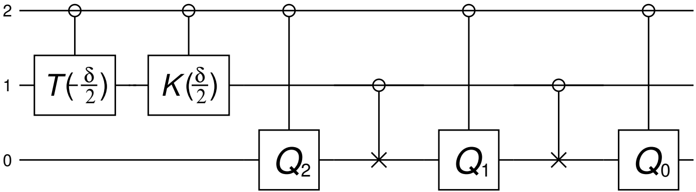

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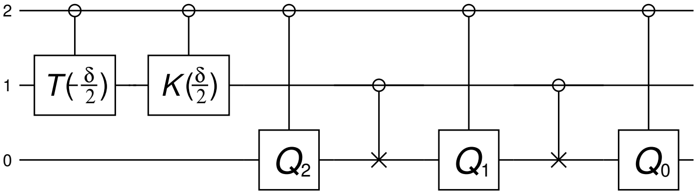

This circuit can be expanded in terms of the general phase shift and rotation gates plus C_{not}, however it requires 25 single qubit gates and 12 C_{not} gates

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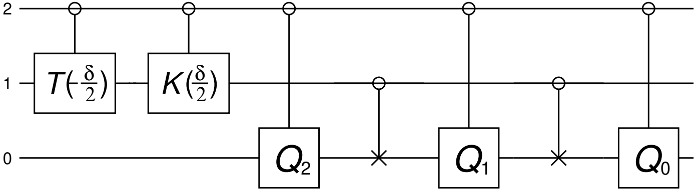

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For a general k-qubit controlled arbitrary gate, one needs 5^k single qubit gates plus $\frac{1}{2}(5^k - 1)$ C_{not} gates which is not the most efficient implementation

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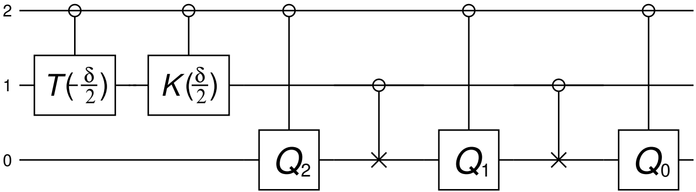

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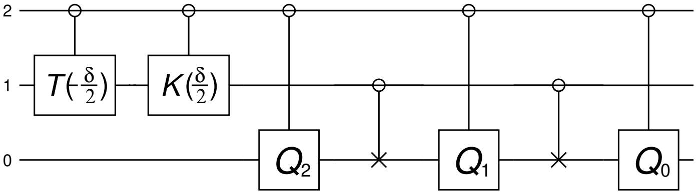

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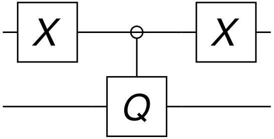
Suppose we want to apply a transformation when the control qubit is 0 or a specific combination of 1's and 0's

This is possible by adding two X gates to the control bit



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These multiply controlled qubit gates will permit arbitrary circuits

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Suppose we have a $(k+1)$-qubit system to which we wish to apply transformation Q on the i^{th} qubit when all the other qubits are in a specific basis state

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It will be useful to define two different transformations using a k-qubit string and a single qubit transformation, Q, on a separate qubit, both of which can be represented as $\bigwedge_x^i Q$

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This operator has the property that: $\bigwedge_{\hat{x}}^i Q = \bigwedge_x^i \hat{Q} = \bigwedge_x^i XQX$

Example 5.4.1

A simplified example of the general $\bigwedge_x^i Q$ transformation is that of a 2-qubit system $|b_1 b_0\rangle$

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	$\|01\rangle$	$I\|b_0\rangle$	$\|01\rangle$	
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Operator	Initial State	Action	Final State	Overall Effect
$\bigwedge_{10}^0 X$	$\|00\rangle$	$I\|b_0\rangle$	$\|00\rangle$	$C_{not}:\ \|b_1\rangle_{ctl} \rightarrow \|b_0\rangle_{tgt}$
	$\|01\rangle$	$I\|b_0\rangle$	$\|01\rangle$	
	$\|10\rangle$	$X\|b_0\rangle$	$\|11\rangle$	
	$\|11\rangle$	$XXX\|b_0\rangle$	$\|10\rangle$	
$\bigwedge_{11}^0 X$	$\|00\rangle$	$I\|b_0\rangle$	$\|00\rangle$	$C_{not}:\ \|b_1\rangle_{ctl} \rightarrow \|b_0\rangle_{tgt}$
	$\|01\rangle$	$I\|b_0\rangle$	$\|01\rangle$	
	$\|10\rangle$	$XXX\|b_0\rangle$	$\|11\rangle$	
	$\|11\rangle$	$X\|b_0\rangle$	$\|10\rangle$	
$\bigwedge_{00}^0 X$	$\|00\rangle$	$X\|b_0\rangle$	$\|01\rangle$	
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Note that $\bigwedge_{01}^1 X$ has the effect of $C_{not}: |b_0\rangle_{ctl} \to |b_1\rangle_{tgt}$

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Applying this operator is identical to applying $\bigwedge_x^j V_2$ where $x = x_{N-2}$ and $j = j_{N-2}$

Generating the general unitary operator

Given the unitary matrix U_{m-1}, and the basis $\{|x_0\rangle, \ldots, |x_{m-1}\rangle, \ldots, |x_{N-1}\rangle\}$, the basis vector $|x_{m-1}\rangle$ is the first on which the operator has a non-trivial action since the identity matrix is $(m-1) \times (m-1)$ and $V_{N-(m-1)}$ mixes the last $N-(m-1)$ basis vectors

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W_m is defined iteratively starting by rewriting $|v_{m-1}\rangle$ as

$$|v_{m-1}\rangle = a_{m-1}|x_{m-1}\rangle + \cdots + c_{N-2}\cos(\theta_{N-2})e^{i\phi_{N-2}}|x_{N-2}\rangle + c_{N-2}\sin(\theta_{N-1})|x_{N-1}\rangle$$

Generating the general unitary operator (cont.)

$$\begin{aligned}|v_{m-1}\rangle &= a_{m-1}|x_{m-1}\rangle + \cdots + a_{N-1}|x_{N-1}\rangle \\ &= a_{m-1}|x_{m-1}\rangle + \cdots + c_{N-2}\cos(\theta_{N-2})e^{i\phi_{N-2}}|x_{N-2}\rangle + c_{N-2}\sin(\theta_{N-1})|x_{N-1}\rangle\end{aligned}$$

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$$a_{N-2} = |a_{N-2}|e^{i\phi_{N-2}}$$

$$c_{N-2} = \sqrt{|a_{N-2}|^2 + |a_{N-1}|^2}$$

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$$= a_{m-1}|x_{m-1}\rangle + \cdots + c_{N-2}\cos(\theta_{N-2})e^{i\phi_{N-2}}|x_{N-2}\rangle + c_{N-2}\sin(\theta_{N-1})|x_{N-1}\rangle$$

$$a_{N-2} = |a_{N-2}|e^{i\phi_{N-2}} \qquad \cos(\theta_{N-2}) = \frac{|a_{N-2}|}{c_{N-2}}$$

$$c_{N-2} = \sqrt{|a_{N-2}|^2 + |a_{N-1}|^2} \qquad \sin(\theta_{N-2}) = \frac{|a_{N-1}|}{c_{N-2}}$$

Generating the general unitary operator (cont.)

$$|v_{m-1}\rangle = a_{m-1}|x_{m-1}\rangle + \cdots + a_{N-1}|x_{N-1}\rangle$$
$$= a_{m-1}|x_{m-1}\rangle + \cdots + c_{N-2}\cos(\theta_{N-2})e^{i\phi_{N-2}}|x_{N-2}\rangle + c_{N-2}\sin(\theta_{N-1})|x_{N-1}\rangle$$

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With these definitions, we can write a multiply controlled set of single qubit operators that acts on $|v_{m-1}\rangle$ to eliminate the $|x_{N-1}\rangle$ term

Generating the general unitary operator (cont.)

$$|v_{m-1}\rangle = a_{m-1}|x_{m-1}\rangle + \cdots + a_{N-1}|x_{N-1}\rangle$$
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$$\bigwedge_{x_{N-2}}^{j_{N-2}} R(\theta_{N-2}) \bigwedge_{x_{N-2}}^{j_{N-2}} K(-\phi_{N-2})|v_{m-1}\rangle = a_{m-1}|x_{m-1}\rangle + \cdots + a'_{N-2}|x_{N-2}\rangle, \quad a'_{N-2} = c_{N-2}$$

Generating the general unitary operator (cont.)

$$
\begin{aligned}
|v_{m-1}\rangle &= a_{m-1}|x_{m-1}\rangle + \cdots + a_{N-1}|x_{N-1}\rangle \\
&= a_{m-1}|x_{m-1}\rangle + \cdots + c_{N-2}\cos(\theta_{N-2})e^{i\phi_{N-2}}|x_{N-2}\rangle + c_{N-2}\sin(\theta_{N-1})|x_{N-1}\rangle
\end{aligned}
$$

$$
a_{N-2} = |a_{N-2}|e^{i\phi_{N-2}} \qquad \cos(\theta_{N-2}) = \frac{|a_{N-2}|}{c_{N-2}}
$$

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The $K(-\phi_{N-2})$ eliminates the phase factor in front of $|x_{N-2}\rangle$

Generating the general unitary operator (cont.)

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The $K(-\phi_{N-2})$ eliminates the phase factor in front of $|x_{N-2}\rangle$ and the $R(\theta_{N-2})$ rotates amplitude from $|x_{N-1}\rangle$ to $|x_{N-2}\rangle$

Generating the general unitary operator (cont.)

The multiply controlled gate ensures that only the two basis vectors with the identical qubit pattern B_{N-2} are affected by this transformation

Generating the general unitary operator (cont.)

The multiply controlled gate ensures that only the two basis vectors with the identical qubit pattern B_{N-2} are affected by this transformation

This same procedure is repeated for the next two lowest order qubit states until $|v_{m-1}\rangle = a'_m|x_{m-1}\rangle \equiv |x_{m-1}\rangle$ and this results in a composite operator

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$$W_m = \bigwedge_{x_{m-1}}^{j_{m-1}} R(\theta_{m-1}) \bigwedge_{x_{m-1}}^{j_{m-1}} K(-\phi_{m-1}) \cdots \bigwedge_{x_{N-2}}^{j_{N-2}} R(\theta_{N-2}) \bigwedge_{x_{N-2}}^{j_{N-2}} K(-\phi_{N-2})$$

Generating the general unitary operator (cont.)

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$$a_i = |a_i| e^{i\phi_i},$$

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This procedure guarantees a general unitary transformation but it is exponentially expensive and therefore is of limited value

Generating the general unitary operator (cont.)

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This procedure guarantees a general unitary transformation but it is exponentially expensive and therefore is of limited value

Making a practical quantum computer requires a more clever approach to take advantage of the inherent efficiency in the computations

A 3-bit example

Consider a 3-qubit system where we wish to establish a Grey code basis

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$$\{ \qquad \}$$

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Consider a 3-qubit system where we wish to establish a Grey code basis

$$\{ \quad |111\rangle, \quad |011\rangle, \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \}$$

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$$\{ \quad |111\rangle, \quad |011\rangle, \quad |001\rangle, \qquad\qquad\qquad\qquad\qquad\qquad \}$$

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$$
\begin{array}{ccccccccccc}
\{ & |111\rangle, & |011\rangle, & |001\rangle, & |000\rangle, & |010\rangle, & |110\rangle, & |100\rangle, & |101\rangle & \} \\
\{ & |x_0\rangle, & |x_1\rangle, & |x_2\rangle, & |x_3\rangle, & |x_4\rangle, & |x_5\rangle, & |x_6\rangle, & |x_7\rangle & \}
\end{array}
$$

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In this case, $n = 3$, $N = 2^n = 8$, and $0 \leq m \leq N - 2 = 6$

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Let's look at the U_6 and U_5 operators

A 3-bit example

Consider a 3-qubit system where we wish to establish a Grey code basis

$$\begin{matrix} \{ & |111\rangle, & |011\rangle, & |001\rangle, & |000\rangle, & |010\rangle, & |110\rangle, & |100\rangle, & |101\rangle & \} \\ \{ & |x_0\rangle, & |x_1\rangle, & |x_2\rangle, & |x_3\rangle, & |x_4\rangle, & |x_5\rangle, & |x_6\rangle, & |x_7\rangle & \} \end{matrix}$$

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Let's look at the U_6 and U_5 operators

$$U_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$

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$$U_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & c & d \end{pmatrix} \qquad U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$

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$$U_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & c & d \end{pmatrix} \qquad U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \end{pmatrix}$$

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In this case, $n = 3$, $N = 2^n = 8$, and $0 \leq m \leq N - 2 = 6$

Let's look at the U_6 and U_5 operators

$$U_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & c & d \end{pmatrix} \qquad U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & o & p & q \\ 0 & 0 & 0 & 0 & 0 & r & s & t \\ 0 & 0 & 0 & 0 & 0 & u & v & w \end{pmatrix}$$

A 3-bit example (cont.)

Our goal is to generate a universal operator

$$U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & o & p & q \\ 0 & 0 & 0 & 0 & 0 & r & s & t \\ 0 & 0 & 0 & 0 & 0 & u & v & w \end{pmatrix}$$

A 3-bit example (cont.)

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$$U = U_0 = C_1 \cdots C_6 U_6$$

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It can be shown that simply combining these two rational rotations $V = P_{\frac{\pi}{4}}S$ gives an irrational rotation

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$T(\alpha)$ rotates about the z-axis and $R(\beta)$ rotates about the y-axis

Making approximating set of gates (cont.)

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This is one possible set, others also exist