Phase shift analysis

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- Review of partial wave analysis

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- Phase shift - partial wave equivalence

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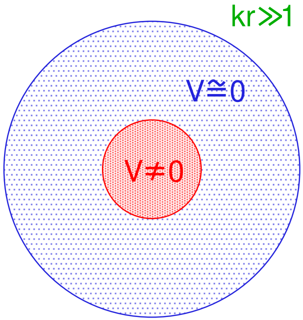

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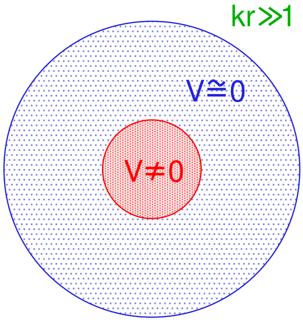

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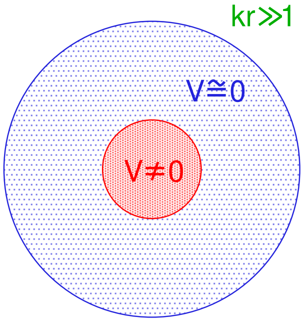

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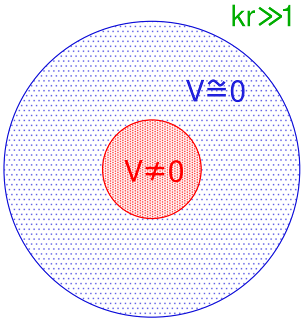

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Scattering region - no approximations applied must solve full potential

Partial wave solutions

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Scattering region – include full potential and match to Intermediate region with incoming plane wave expanded in partial waves

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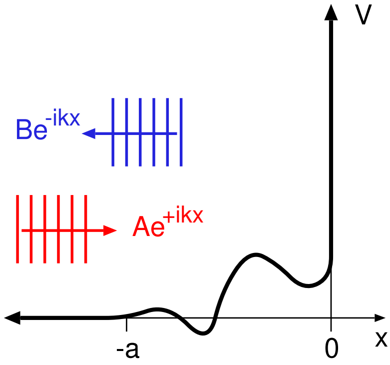
We can show how the phase shift approach is applied to a 1D case, then the more general 3D case where it is equivalent to partial wave description

Phase shifts in 1-D

For a 1D system with a solid "wall" at $x = 0$, we can write the incident



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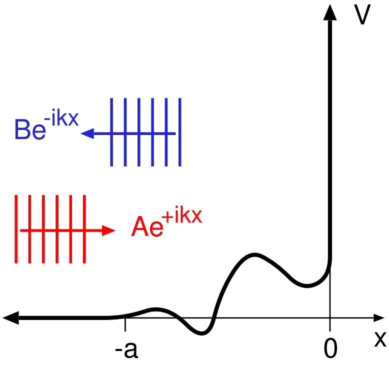

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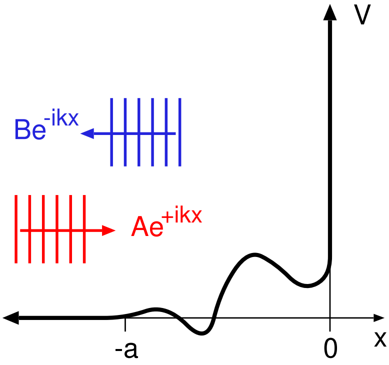

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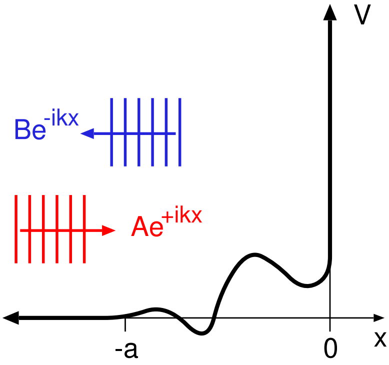

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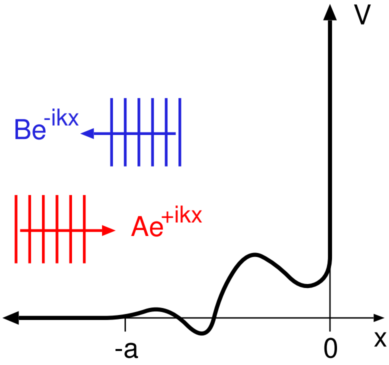

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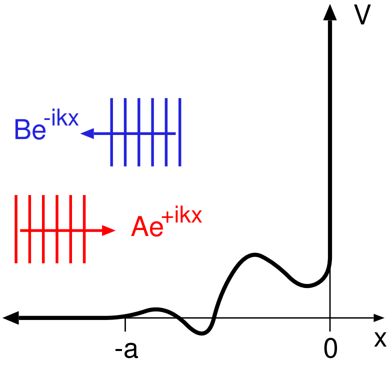

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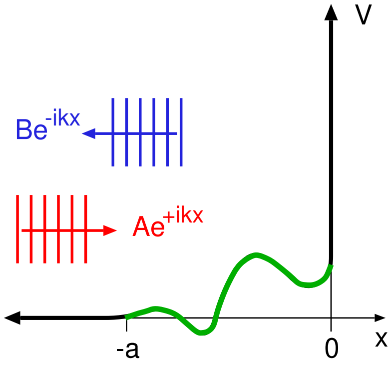

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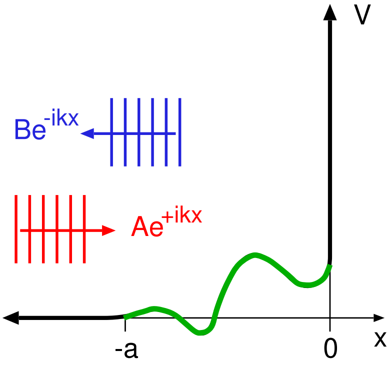

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The integral Schrödinger equation

The integral Schrödinger equation

- Development of the integral equation

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- Development of the integral equation
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- Development of the integral equation
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- Integrating the Green's function

Integral form of the Schrödinger equation

Starting with the time-independent Schrödinger equation

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$$g(\vec{s}) = \frac{1}{(2\pi)^{3/2}(k^2 - s^2)}$$

Green's functions

$G(\vec{r})$ is a Green's function and represents the response of a linear differential equation to a delta function source

by determining the Green's function, we can solve the differential equation's response to an arbitrary source using a simple integral equation

the task is to solve the delta function source equation for the Green's function which can be done by taking a Fourier transform

$$\delta^3(\vec{r}) = \left(\nabla^2 + k^2\right) G(\vec{r})$$

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Integrating the Green's function

$$G(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}} \frac{1}{(k^2 - s^2)}\, d\vec{s}$$

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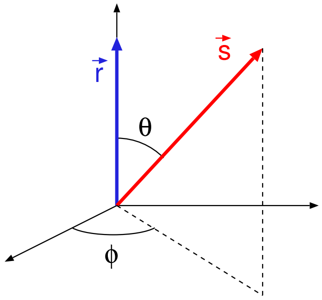

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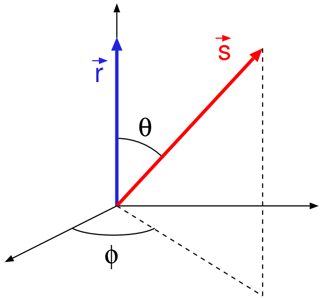

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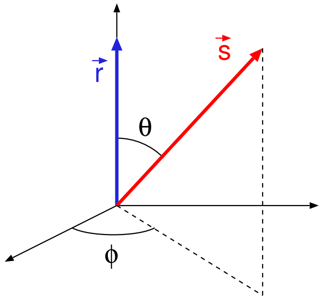

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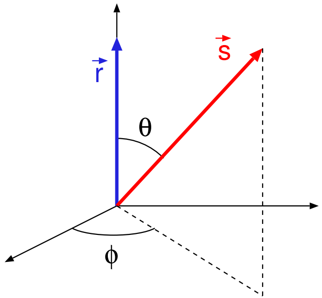

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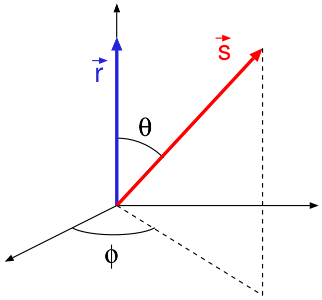

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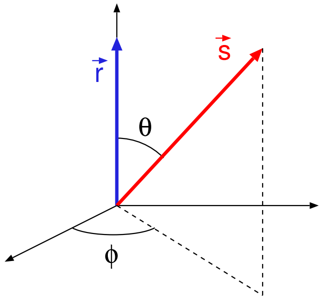

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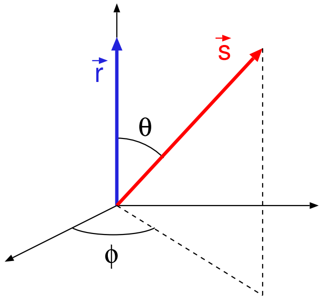

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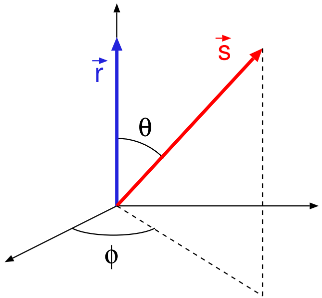

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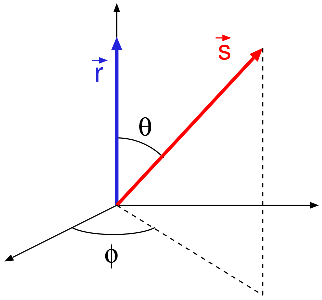

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this integral needs to be perfomed using Cauchy's formula

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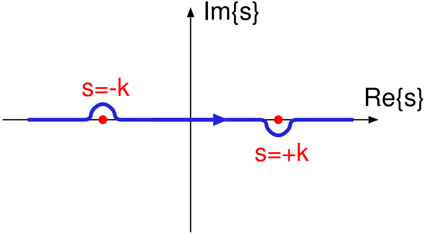
Contour integration

$$\oint \frac{f(z)}{(z-z_0)}\,dz = 2\pi i f(z_0)$$

Contour integration



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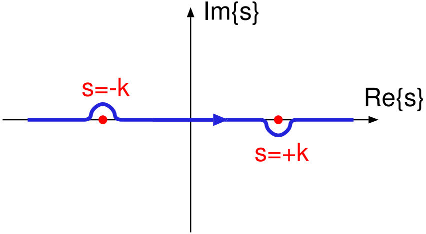

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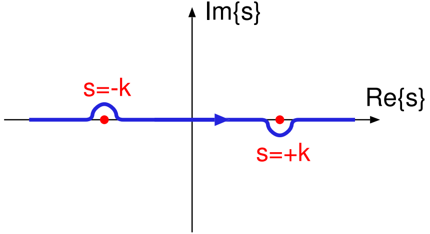
deform the path to loop around the negative pole in the positive direction by an infinitesimal amount, and the positive pole in the negative direction

Contour integration



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close the contour at $Re\{s\} \to \pm\infty$ in a semi-circle such that $|s| \to \infty$

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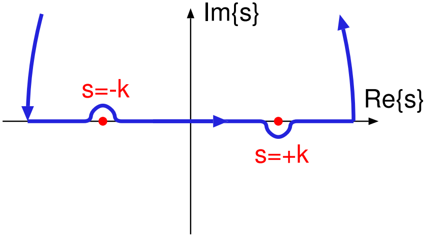

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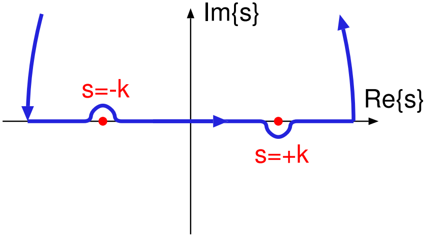

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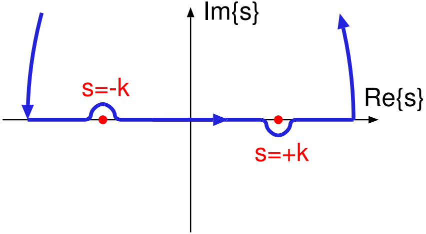

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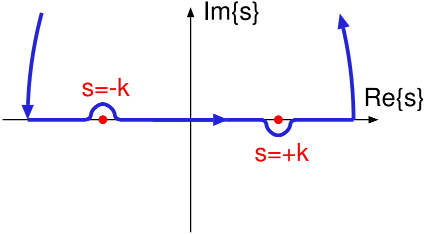

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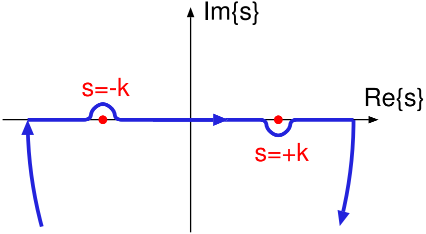

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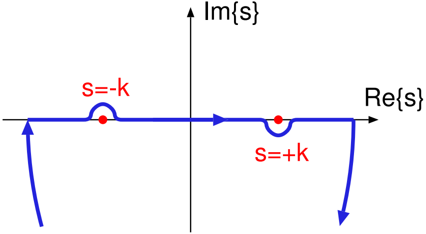

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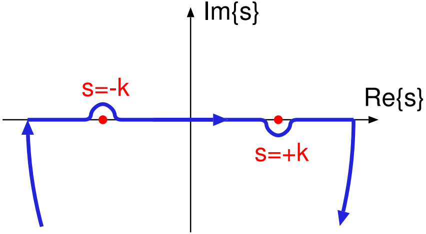

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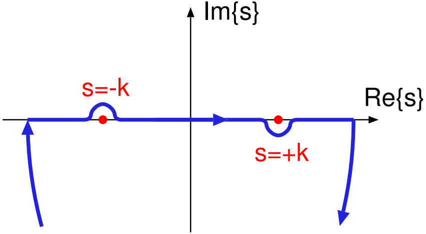

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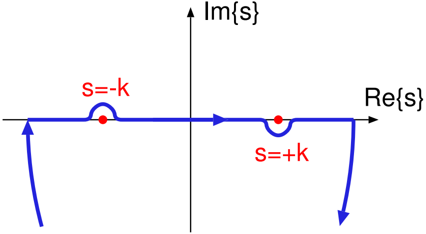
$$I_2 = -\oint \left[\frac{s\, e^{-isr}}{s-k}\right] \frac{1}{s+k} ds$$
$$= -2\pi i \left[\frac{s\, e^{-isr}}{s-k}\right|_{s=-k} = -i\pi\, e^{ikr}$$

Contour integration



$$\oint \frac{f(z)}{(z-z_0)}\,dz = 2\pi i f(z_0)$$

$$I_1 = \oint \left[\frac{s\,e^{isr}}{s+k}\right]\frac{1}{s-k}\,ds$$

$$= 2\pi i \left[\frac{s\,e^{isr}}{s+k}\right|_{s=k} = i\pi\,e^{ikr}$$

$$I_2 = -\oint \left[\frac{s\,e^{-isr}}{s-k}\right]\frac{1}{s+k}\,ds$$

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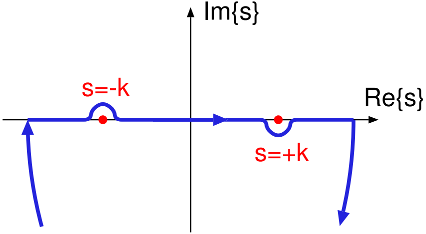
deform the path to loop around the negative pole in the positive direction by an infinitesimal amount, and the positive pole in the negative direction

close the contour at $Re\{s\} \to \pm\infty$ in a semi-circle such that $|s| \to \infty$

$$G(\vec{r}) = \frac{i}{8\pi^2 r}\left[\left(i\pi\,e^{ikr}\right) - \left(-i\pi\,e^{ikr}\right)\right]$$

Contour integration



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$$I_2 = -\oint \left[\frac{s\, e^{-isr}}{s-k}\right] \frac{1}{s+k} ds$$

$$= -2\pi i \left[\frac{s\, e^{-isr}}{s-k}\right|_{s=-k} = -i\pi\, e^{ikr}$$

$$G(\vec{r}) = \frac{i}{8\pi^2 r}\left[\left(i\pi\, e^{ikr}\right) - \left(-i\pi\, e^{ikr}\right)\right] = -\frac{e^{ikr}}{4\pi r}$$