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- Aharonov-Bohm effect

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Homework Assignment #10:
Chapter 10:2,4,5,7,9,12
due Tuesday, April 09, 2019

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Homework Assignment #11:
Chapter 10:14,18,20; 11:2,4,6
due Tuesday, April 16, 2019

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Tuesday, April 23, 2019

Aharonov-Bohm effect

THE

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Significance of Electromagnetic Potentials in the Quantum Theory

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In this paper, we discuss some interesting properties of the electromagnetic potentials in the quantum domain. We shall show that, contrary to the conclusions of classical mechanics, there exist effects of potentials on charged particles, even in the region where all the fields (and therefore the forces on the particles) vanish. We shall then discuss possible experiments to test these conclusions; and, finally, we shall suggest further possible developments in the interpretation of the potentials.

1. INTRODUCTION

IN classical electrodynamics, the vector and scalar potentials were first introduced as a convenient mathematical aid for calculating the fields. It is true that in order to obtain a classical canonical formalism, the potentials are needed. Nevertheless, the fundamental equations of motion can always be expressed directly in terms of the fields alone.

In the quantum mechanics, however, the canonical formalism is necessary, and as a result, the potentials cannot be eliminated from the basic equations. Nevertheless, these equations, as well as the physical quantities, are all gauge invariant; so that it may seem that even in quantum mechanics, the potentials themselves have no independent significance.

In this paper, we shall show that the above conclusions are not correct and that a further interpretation of the potentials is needed in the quantum mechanics.

2. POSSIBLE EXPERIMENTS DEMONSTRATING THE ROLE OF POTENTIALS IN THE QUANTUM THEORY

In this section, we shall discuss several possible experiments which demonstrate the significance of potentials in the quantum theory. We shall begin with a simple example.

Suppose we have a charged particle inside a "Faraday cage" connected to an external generator which causes the potential on the cage to alternate in time. This will add to the Hamiltonian of the particle a term $V(x,t)$ which is, for the region inside the cage, a function of time only. In the nonrelativistic limit (and we shall assume this almost everywhere in the following discussions) we have, for the region inside the cage, $H=H_0+V(t)$ where H_0 is the Hamiltonian when the generator is not functioning, and $V(t)=e\phi(t)$. If $\psi_0(x,t)$ is a solution of the Hamiltonian H_0, then the solution for H will be

$$\psi=\psi_0 e^{-iS/\hbar}, \quad S=\int V(t)dt,$$

which follows from

$$i\hbar\frac{\partial\psi}{\partial t}=\left(i\hbar\frac{\partial\psi_0}{\partial t}+\psi_0\frac{\partial S}{\partial t}\right)e^{-iS/\hbar}=[H_0+V(t)]\psi=H\psi.$$

The new solution differs from the old one just by a phase factor and this corresponds, of course, to no change in any physical result.

Now consider a more complex experiment in which a single coherent electron beam is split into two parts and each part is then allowed to enter a long cylindrical metal tube, as shown in Fig. 1.

After the beams pass through the tubes, they are combined to interfere coherently at F. By means of time-determining electrical "shutters" the beam is chopped into wave packets that are long compared with the wavelength λ, but short compared with the length of the tubes. The potential in each tube is determined by a time delay mechanism in such a way that the potential is zero in region I (until each packet is well inside its tube). The potential then grows as a function of time, but differently in each tube. Finally, it falls back to zero, before the electron comes near the

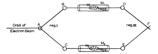

FIG. 1. Schematic experiment to demonstrate interference with time-dependent scalar potential. A, B, C, D, E: suitable devices to separate and divert beams. W_1, W_2: wave packets. M_1, M_2: cylindrical metal tubes. F: interference region.

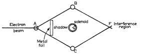

FIG. 2. Schematic experiment to demonstrate interference with time-independent vector potential.

other edge of the tube. Thus the potential is nonzero only while the electrons are well inside the tube (region II). When the electron is in region III, there is again no potential. The purpose of this arrangement is to ensure that the electron is in a time-varying potential without ever being in a field (because the field does not penetrate far from the edges of the tubes, and is nonzero only at times when the electron is far from these edges).

Now let $\psi(x,t)=\psi_1^0(x,t)+\psi_2^0(x,t)$ be the wave function when the potential is absent (ψ_1^0 and ψ_2^0 representing the parts that pass through tubes 1 and 2, respectively). But since V is a function only of t wherever ψ is appreciable, the problem for each tube is essentially the same as that of the Faraday cage. The solution is then

$$\psi=\psi_1^0 e^{-iS_1/\hbar}+\psi_2^0 e^{-iS_2/\hbar},$$

where

$$S_1=e\int\varphi_1 dt, \quad S_2=e\int\varphi_2 dt.$$

It is evident that the interference of the two parts at F will depend on the phase difference $(S_1-S_2)/\hbar$. Thus, there is a physical effect of the potentials even though no force is ever actually exerted on the electron. The effect is evidently essentially quantum-mechanical in nature because it comes in the phenomenon of interference. We are therefore not surprised that it does not appear in classical mechanics.

From relativistic considerations, it is easily seen that the covariance of the above conclusion demands that there should be similar results involving the vector potential, **A**.

The phase difference, $(S_1-S_2)/\hbar$, can also be expressed as the integral $(e/\hbar)\oint\varphi dt$ around a closed circuit in space-time, where φ is evaluated at the place of the center of the wave packet. The relativistic generalization of the above integral is

$$\frac{e}{\hbar}\oint\left(\varphi dt-\frac{\mathbf{A}}{c}\cdot d\mathbf{x}\right),$$

where the path of integration now goes over any closed circuit in space-time.

As another special case, let us now consider a path in space only (t=constant). The above argument suggests that the associated phase shift of the electron wave function ought to be

$$\Delta S/\hbar=-\frac{e}{c\hbar}\oint\mathbf{A}\cdot d\mathbf{x},$$

where $\oint\mathbf{A}\cdot d\mathbf{x}=\int\mathbf{H}\cdot d\mathbf{s}=\phi$ (the total magnetic flux inside the circuit).

This corresponds to another experimental situation. By means of a current flowing through a very closely wound cylindrical solenoid of radius R, center at the origin and axis in the z direction, we create a magnetic field, **H**, which is essentially confined within the solenoid. However, the vector potential, **A**, evidently, cannot be zero everywhere outside the solenoid, because the total flux through every circuit containing the origin is equal to a constant

$$\phi_0=\int\mathbf{H}\cdot d\mathbf{s}=\int\mathbf{A}\cdot d\mathbf{x}.$$

To demonstrate the effects of the total flux, we begin, as before, with a coherent beam of electrons. (But now there is no need to make wave packets.) The beam is split into two parts, each going on opposite sides of the solenoid, but avoiding it. (The solenoid can be shielded from the electron beam by a thin plate which casts a shadow.) As in the former example, the beams are brought together at F (Fig. 2).

The Hamiltonian for this case is

$$H=\frac{[\mathbf{P}-(e/c)\mathbf{A}]^2}{2m}.$$

In singly connected regions, where $\mathbf{H}=\nabla\times\mathbf{A}=0$, we can always obtain a solution for the above Hamiltonian by taking $\psi=\psi_0 e^{-iS/\hbar}$, where ψ_0 is the solution when $\mathbf{A}=0$ and where $\nabla S/\hbar=(e/c)\mathbf{A}$. But, in the experiment discussed above, in which we have a multiply connected region (the region outside the solenoid), $\psi_0 e^{-iS/\hbar}$ is a non-single-valued function[1] and therefore, in general, not a permissible solution of Schrödinger's equation. Nevertheless, in our problem it is still possible to use such solutions because the wave function splits into two parts $\psi=\psi_1+\psi_2$, where ψ_1 represents the beam on

[1] Unless $\phi_0=nhc/e$, where n is an integer.

"Significance of electromagnetic potentials in quantum theory," Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485-491 (1959).

Aharonov-Bohm theory

Consider the more general case where a particle moves through a region where $\vec{B} = \nabla \times \vec{A} = 0$ but $\vec{A} \neq 0$

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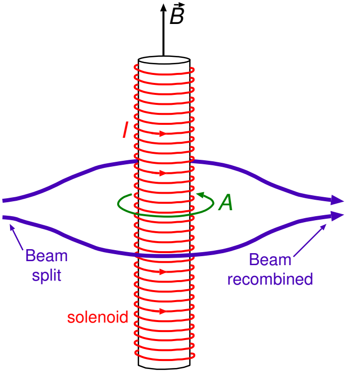

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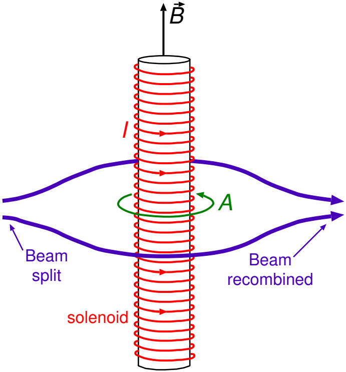

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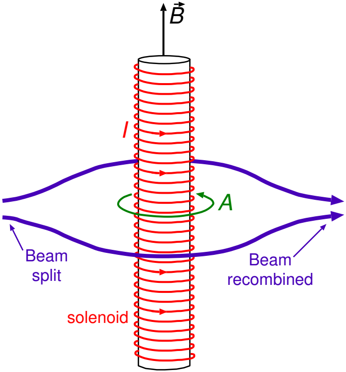

This is a so-called non-holonomic process which involves Berry's Phase

Aharonov-Bohm experiment

the mass spectrographic analyses and Dr. M. E. Norberg of the Corning Glass Company for providing us with porous Vycor glass.

*This work was supported by the Atomic Energy Commission and, in the case of one of the authors (F. R.), also by the Alfred P. Sloan Foundation.

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[6]More exactly, $D_{i3} \propto (M + M^2/M_3)^{-1/4}$, where M_3 is the effective mass of a He^3 atom. A reasonable estimate is $M_3 = 2M_{He}$.
[7]This is deduced from the temperature range from about 0.6 to 0.5°K where the mobility differences are sufficiently large for the subtraction analysis to be feasible.
[8]I. M. Khalatnikov and V. N. Zharkov, J. Exptl. Theoret. Phys. U.S.S.R. 32, 1108 (1957) [translation: Soviet Phys.-JETP 5, 905 (1957)].

SHIFT OF AN ELECTRON INTERFERENCE PATTERN BY ENCLOSED MAGNETIC FLUX

R. G. Chambers
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(Received May 27, 1960)

Aharonov and Bohm[1] have recently drawn attention to a remarkable prediction from quantum theory. According to this, the fringe pattern in an electron interference experiment should be shifted by altering the amount of magnetic flux passing between the two beams (e.g., in region a of Fig. 1), even though the beams themselves pass only through field-free regions. Theory predicts a shift of n fringes for an enclosed flux Φ of nhc/e; it is convenient to refer to a natural "flux unit," $hc/e = 4.135 \times 10^{-7}$ gauss cm^2. It has since been pointed out[2] that the same conclusion had previously been reached by Ehrenberg and Siday,[3] using semiclassical arguments, but these authors perhaps did not sufficiently stress the remarkable nature of the result, and their work appears to have attracted little attention.

Clearly the first problem to consider, experimentally, is the effect on the fringe system of stray fields not localized to region a but extending, e.g., over region a' in Fig. 1. In addition to the "quantum" fringe shift due to the enclosed flux, there will then be a shift due simply to curvature of the electron trajectories by the field. A straightforward calculation shows that in a "biprism" experiment,[4] such a field should produce a fringe displacement which exactly keeps pace with the deflection of the beams by the field, so that the fringe system appears to remain undisplaced relative to the envelope of the pattern. A field of type a, on the other hand, should leave the envelope undisplaced, and produce a fringe shift within it. In the Marton[5] interferometer, conditions are different, and a field of type a' should leave the fringes undisplaced in space. This explains how Marton et al.[5] were able to observe fringes in the presence of stray 60-cps fields probably large enough to have destroyed them otherwise; this experiment thus constitutes an inadvertent check of the existence of the "quantum" shift.[2]

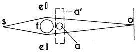


FIG. 1. Schematic diagram of interferometer, with source s, observing plane o, biprism e, f, and confined and extended field regions a and a'.

To obtain a more direct check, a Philips EM100 electron microscope[6] has been modified so that it can be switched at will from normal operation to operation as an interferometer. Fringes are produced by an electrostatic "biprism" consisting of an aluminized quartz fiber f (Fig. 1) flanked by two earthed metal plates e; altering the positive potential applied to f alters the effective angle of the biprism.[4] The distances s-f and f-o (Fig. 1) are about 6.7 cm and 13.4 cm, respectively. With this microscope it was not possible to reduce the virtual source diameter below about 0.2 μ, so that it was necessary to use a fiber f only about 1.5 μ in diameter and a

VOLUME 5, NUMBER 1 PHYSICAL REVIEW LETTERS JULY 1, 1960

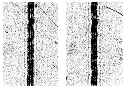


FIG. 2. (a) Fringe pattern due to biprism alone. (b) Pattern displaced by 2.5 fringe widths by field of type a'.

very small biprism angle, to produce a wide pattern of fringes which would not be blurred out by the finite source size. The fringe pattern obtained is shown in Fig. 2(a); the fringe width in the observing plane o is about 0.6 μ.

We first examined the effect of a field of type a', produced by a Helmholtz pair of single turns 3 mm in diameter just behind the biprism. Fields up to 0.3 gauss were applied, sufficient to displace the pattern by up to 30 fringe widths, and as predicted the appearance of the pattern was completely unchanged. Figure 2(b), for instance, shows the pattern in a field producing a displacement of about 2.5 fringe widths. In the absence of the "quantum" shift due to the enclosed flux, this pattern would have had the light and dark fringes interchanged. We also verified that with this interferometer, unlike Marton's, a small ac field suffices to blur out the fringe system completely. These results confirm the presence of the quantum shift in fields of type a'.

Of more interest is the effect predicted for a field of type a, where intuition might expect no effect. Such a field was produced by an iron whisker,[7] about 1 μ in diameter and 0.5 mm long, placed in the shadow of the fiber f. Whiskers as thin as this are expected theoretically[8] and found experimentally[9] to be single magnetic domains; moreover they are found to taper[9] with a slope of the order of 10^{-3}, which is extremely convenient for the present purpose. An iron whisker 1 μ in diameter will contain about 400 flux units; if it tapers uniformly with a slope of 10^{-3}, the flux content will change along the length at a rate $d\Phi/dz$ of about 1 flux unit per micron. Thus if such a whisker is placed in position a (Fig. 1), we expect to see a pattern in which the envelope is undisplaced, but the fringe system within the envelope is inclined at an angle of the order of one fringe width per micron. Since the fringe width in the observing plane is 0.6 μ, and there is a "pin-hole" magnification of $\times 3$ between the biprism-fiber assembly and the observing plane, we thus expect the fringes to show a tilt of order 1 in 5 relative to the envelope of the pattern. Precisely this is observed experimentally, as shown in Fig. 3(a). It will be seen that the whisker taper is not uniform, but in this example becomes very small in the upper part of the picture.

In fact the biprism is an unnecessary refinement for this experiment: Fresnel diffraction into the shadow of the whisker is strong enough to produce a clear fringe pattern from the whisker alone. Thus Fig. 3(b) shows the same section of whisker as Fig. 3(a), moved just out of the shadow of the biprism fiber. The biprism fringes are now unperturbed; the Fresnel fringes in the shadow of the whisker show exactly the same pattern of fringe shifts along their length as in Fig. 3(a). Figure 3(c) shows a further example of these fringes, from a different part of the same whisker, with the biprism moved out of the way. The whisker here is tapering more rapidly.

These fringe shifts cannot be attributed to direct interaction between the electrons and the surface of the whisker, since in Fig. 3(a) the whisker

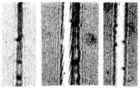
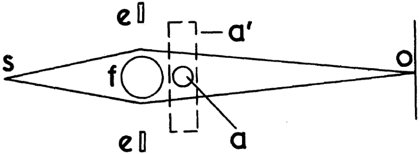


FIG. 3. (a) Tilted fringes produced by tapering whisker in shadow of biprism fiber. (b) Fresnel fringes in the shadow of the whisker itself, just outside shadow of fiber. (c) Same as (b), but from a different part of the whisker, and with fiber out of the field of view.

"Shift of an electron interference pattern by enclosed magnetic flux," R.G. Chambers, *Phys. Rev. Lett.* **5**, 3-5 (1960).

Aharonov-Bohm experiment



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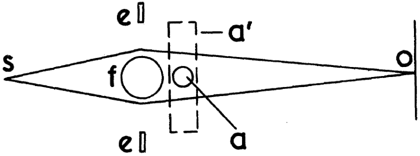


s: electron source

o: observing plane

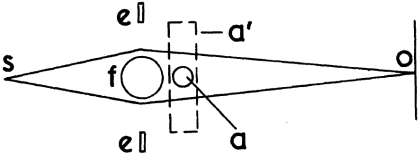
e, f: biprism

a: confined field region

a': extended field region

"Shift of an electron interference pattern by enclosed magnetic flux," R.G. Chambers, *Phys. Rev. Lett.* **5**, 3-5 (1960).

Aharonov-Bohm experiment



s: electron source

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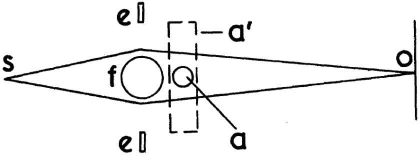
a: confined field region

a′: extended field region

what about the effect of stray fields in region *a*′ which can curve the electron beams electrostatically?

"Shift of an electron interference pattern by enclosed magnetic flux," R.G. Chambers, *Phys. Rev. Lett.* **5**, 3-5 (1960).

Aharonov-Bohm experiment



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what about the effect of stray fields in region a' which can curve the electron beams electrostatically?

in the biprism, the quantum effect exactly cancels the stray field leaving the interference pattern unchanged

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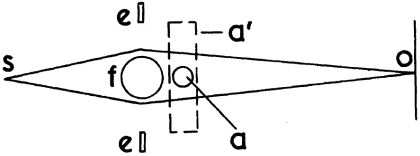

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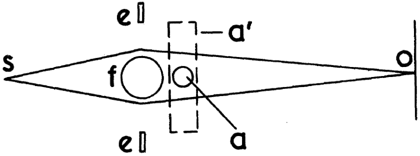

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modified electron microscope,

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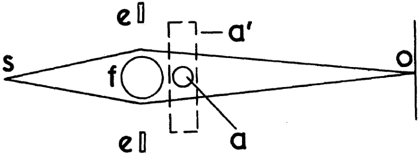

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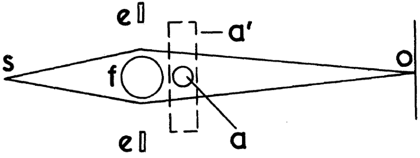
Aharonov-Bohm experiment



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modified electron microscope, biprism consists of an aluminized quartz fiber (f) and two grounded metal plates (e), Aharonov-Bohm effect will produce a shift of n fringes for $\Phi = nhc/e$

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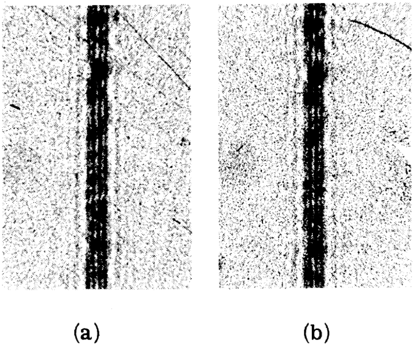
Aharonov-Bohm experiment



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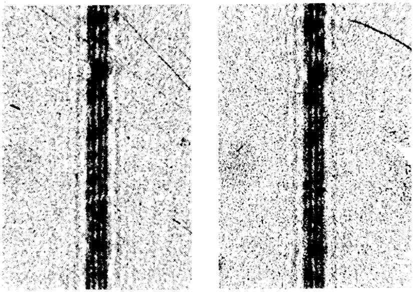

(a) (b)

(a) is with no additional field applied in extended region

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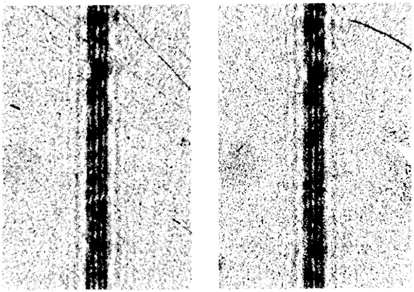
Aharonov-Bohm experiment



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(b) has 25mG, which alone would invert the fringe, applied with no visible fringe shift

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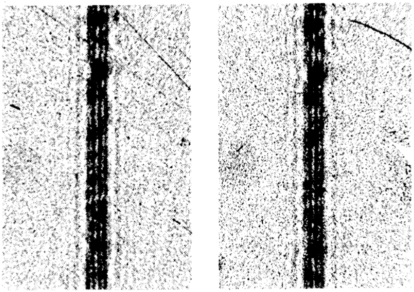

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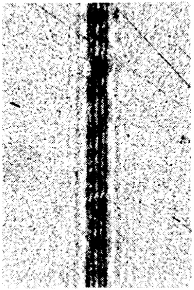
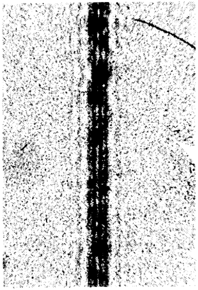

(a)


(b)

(a) is with no additional field applied in extended region

(b) has 25mG, which alone would invert the fringe, applied with no visible fringe shift

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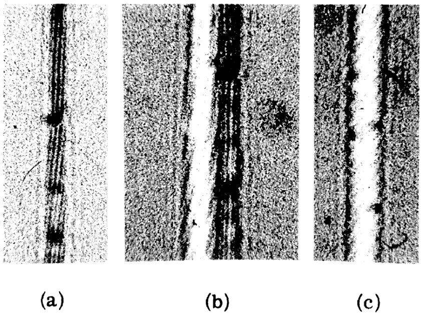
This calibration experiment shows that the Aharonov-Bhom effect is present and balances the electrostatic fringe shifts in a region where there is both a flux AND a field

"Shift of an electron interference pattern by enclosed magnetic flux," R.G. Chambers, *Phys. Rev. Lett.* **5**, 3-5 (1960).

Aharonov-Bohm experiment



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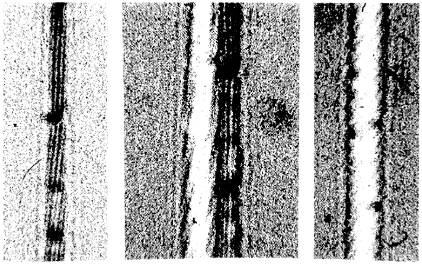
Aharonov-Bohm experiment



(a) (b) (c)

(a) a tapered iron whisker produces a confined field and flux with a gradient along the z-axis manifested in tilted fringes

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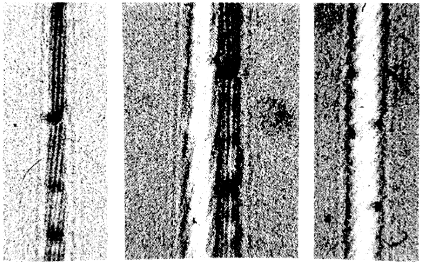

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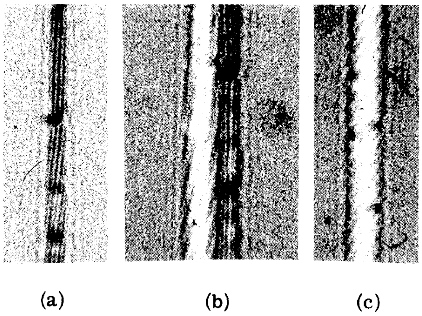

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(c) higher taper, again using Fresnel diffraction in fiber shadow but with biprism removed shows more highly tilted fringes

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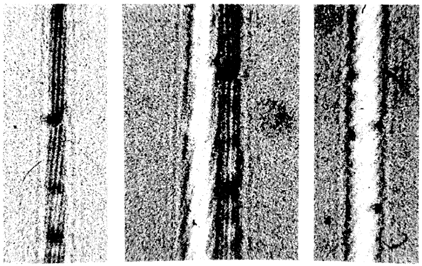

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Chambers says: "I am indebted for Mr. Aharonov and Dr. Bohm for telling me of their work before publication.."

"Shift of an electron interference pattern by enclosed magnetic flux," R.G. Chambers, *Phys. Rev. Lett.* **5**, 3-5 (1960).

A-B effect in a normal metal

Observation of h/e Aharonov-Bohm Oscillations in Normal-Metal Rings

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IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
(Received 27 March 1985)

Magnetoresistance oscillations periodic with respect to the flux h/e have been observed in submicron-diameter Au rings, along with weaker $h/2e$ oscillations. The h/e oscillations persist to very large magnetic fields. The background structure in the magnetoresistance was *not* symmetric about zero field. The temperature dependence of both the amplitude of the oscillations and the background are consistent with the recent theory by Stone.

PACS numbers: 72.15.Gd, 72.90.+y, 73.60.Dt

Electron wave packets circling a magnetic flux should exhibit the phase shift introduced by the magnetic vector potential **A**.[1] In a metallic ring, small enough so that the electron states are not randomized by inelastic (or magnetic) scattering during the traversal of the arm of the ring, an interference pattern should be present in the magnetoresistance of the device.[2] Electrons traveling along one arm will acquire a phase change δ_1, and electrons in the other arm will, in general, suffer a different phase change δ_2. Changing the magnetic flux encircled by the ring will tune the phase change along one arm of the ring by a well-defined amount $\delta_B = (e/\hbar)\int \mathbf{A}\cdot d\mathbf{l}$ and by $-\delta_B$ along the other arm. The phase tuning should appear as cycles of destructive and constructive interference of the wave packets, the period of the cycle being $\Phi_0 = h/e$. This interference should be reflected in the transport properties of the ring as described by Landauer's formula.[2-4] In this Letter, we describe the first experimental observation of the oscillations periodic with respect to Φ_0 in the magnetoresistance of a normal-metal ring.

Interference effects involving the flux h/e have been previously observed in a two-slit interference experiment involving coherent beams of electrons.[5] Magnetoresistance oscillations in single-crystal whiskers of bismuth periodic in h/e have been reported at low fields for the case where the extremum of the Fermi surface is cut off by the sample diameter.[6] Resistance oscillations of period $h/2e$ (flux quantization) have been seen in superconducting cylinders.[7] Four years ago, magnetoresistance oscillations of period $\frac{1}{2}\Phi_0$ were predicted on the basis of weak localization in multiply connected devices.[8] This is the same flux period as observed in superconductors, because of the similarity between the superconductor pairing and the "self-interference" described by the theory of weak localization.[9] Since the first experiment by Sharvin and Sharvin,[10] there have been several observations of the superconducting flux period $\frac{1}{2}\Phi_0$ in normal-metal cylinders and networks of loops.[11] To date, there have been no observations of the one-electron flux period Φ_0, and its existence is controversial. Several recent theoretical papers have argued that the h/e period will be present in strictly one-dimensional rings,[2] and even in rings composed of wires with finite width.[4] Others have claimed that only $h/2e$ oscillations will be observed regardless of device size and topology.[12]

Theoretical work which relies upon ensemble-averaging techniques has uniformly predicted $h/2e$ oscillations[8,12]; calculations of the conductance exclusive of the averaging have predicted h/e oscillations as well.[2,4] The difference between a single ring and a network of rings or a long cylinder is, therefore, crucial. The network of many rings and the long cylinder extend much farther than the distance $[L_\phi = (D\tau_\phi)^{1/2}$, where D is the diffusion constant and τ_ϕ is the time between phase-breaking collisions] that the electron travels before randomly changing its phase. For this reason, it is believed that samples much longer than L_ϕ physically incorporate the ensemble averaging.[4,13] Each section (longer than L_ϕ) of a macroscopic sample is quantum-mechanically independent because the electron states are randomized between the sections. The single mesoscopic ring (diameter $< L_\phi$) does not average in this way because the entire sample is quantum-mechanically coherent.[4,13]

There exists a further complication in normal metals; the magnetic flux penetrates the wires composing the device. Stone[14] has shown that the flux in the wire leads to an *aperiodic* fluctuation in the magnetoresistance. This fluctuation was the main complication in interpreting the earlier experiments[15] where the diameter of the ring was not much larger than the widths of the wires. On the basis of the analysis, a prediction was made that, in a ring having an area much larger than the area covered by the wires, the oscillations would be clearly observed, since the period would then be much smaller than the field scale of the fluctuations.

With this in mind, we constructed several devices each containing a single loop or a lone wire. The samples were drawn with a scanning transmission electron microscope (STEM) on a polycrystalline gold film 38 nm thick having a resistivity $\rho \simeq 5\ \mu\Omega\cdot$cm at $T = 4$ K. The fabrication process has been described previously.[16] A photograph of the larger ring is shown in Fig. 1. Here we will describe the results from two of the

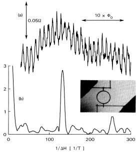

FIG. 1. (a) Magnetoresistance of the ring measured at $T = 0.01$ K. (b) Fourier power spectrum in arbitrary units containing peaks at h/e and $h/2e$. The inset is a photograph of the larger ring. The inside diameter of the loop is 784 nm, and the width of the wires is 41 nm.

rings (average diameters 825 and 245 nm) and a lone wire (length 300 nm). The samples were cooled in the mixing chamber of a dilution refrigerator, and the resistance was measured with a four-probe bridge operated at 205 Hz and 200 nA (rms).

Typical magnetoresistance data from the larger-diameter ring are displayed in Fig. 1(a). Periodic oscillations are clearly visible superimposed on a more slowly varying background. The period of the high-frequency oscillations is $\Delta H = 0.007\,59$ T. This period corresponds to the addition of the flux $\Phi_0 = h/e$ to the area of the hole. From the average area (one half of the sum of the area from the inside diameter and that from the outside diameter) measured with the STEM, $\Phi_0 = 0.007\,80$ T. The area measurement is accurate to within $\simeq 10\%$. As a result of the large aspect ratio, we can say unequivocally that the periodic oscillations are *not* consistent with $h/2e$. They are certainly the single-electron process predicted recently.[2,4] In the Fourier power spectrum [Fig. 1(b)] of these data, two peaks are visible at $1/\Delta H = 131$ and 260 T^{-1} corresponding respectively to h/e and $h/2e$. (Since the h/e oscillations are not strictly sinusoidal, we cannot be certain whether the $h/2e$ peak is the self-interference process or harmonic content in the Φ_0 oscillations.) That the $h/2e$ period is less significant than the h/e period is consistent with the theory for rings which are moderately resistive.[4] We note that the amplitude of the h/e oscillations at the lowest temperatures is about 0.1% of the resistance at $H = 0$, at least a factor of 10

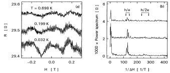

FIG. 2. (a) Magnetoresistance data from the ring in Fig. 1 at several temperatures. (b) The Fourier transform of the data in (a). The data at 0.199 and 0.698 K have been offset for clarity of display. The markers at the top of the figure indicate the bounds for the flux periods h/e and $h/2e$ based on the measured inside and outside diameters of the loop.

larger than the oscillations observed in normal-metal cylinders and networks of loops.[8,10,11]

Figure 2(a) contains resistance data for three temperatures over a larger range of magnetic field. Surprisingly, the oscillations persist to rather higher magnetic field [$H > 8$ T (our largest available field) or over 1000 periods] than expected from estimates which assumed that the phase difference between the inside edge of the ring and the outside edge should completely destroy the periodic effects. The argument that the flux in the metal should destroy the oscillations relies on the simple assumption that the wire consists of parallel but noninteracting conduction paths. If instead the electron path in the wire is sufficiently erratic to "cover" the whole area of the wire, then no phase difference exists between the inside diameter and the outside diameter.[17]

Figure 2(b) contains the Fourier spectra of the data in Fig. 2(a). Again, the fundamental h/e period appears as the large peak at $1/\Delta H = 131$ T^{-1}, and near $1/\Delta H = 260$ T^{-1} there is a small feature in the spectrum. There is also a peak near 5 T^{-1} which is the average field scale of the aperiodic fluctuations.[14] The detailed structure of the h/e peak in the power spectrum is probably the results of mixing of the field scales corresponding to the area of the hole in the ring and the area of the arms of the ring.[18] (The simple difference between inside and outside area implies a splitting of more than 20 T^{-1}, whereas the observed splitting in the peak structure has never been more than 7 T^{-1}.) A simple extension of the multichannel Landauer formula for a ring with flux piercing the arms implies that the Arharonov-Bohm oscillations will be modulated by an aperiodic function.[18] Roughly speaking, the field scale in which the aperiodic function fluctuates is that for the addition of another flux quantum to the arms of the ring. The field scale of the modulating function mixes with the Aharonov-Bohm period to give structure to the peak. As seen in Fig.

"Observation of h/e Aharonov-Bohm oscillations in normal-metal rings," R.A. Webb, et al., *Phys. Rev. Lett.* **54**, 2696-2699 (1985).

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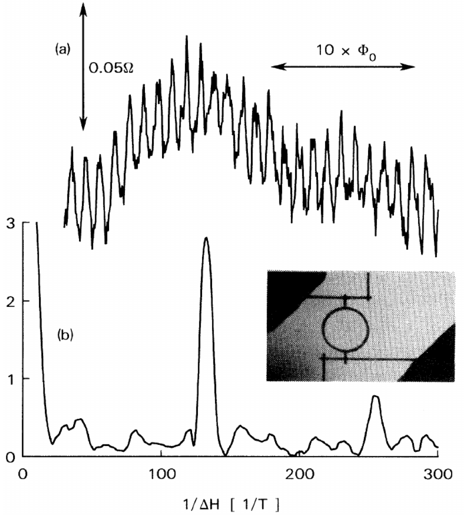

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Electrostatic A-B effect

Magneto-electric Aharonov–Bohm effect in metal rings

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The quantum-mechanical phase of the wavefunction of an electron can be changed by electromagnetic potentials, as was predicted by Aharonov and Bohm[1] in 1959. Experiments on propagating electron waves in vacuum have revealed both the magnetic[2–4] and electrostatic[5] Aharonov–Bohm effect. Surprisingly, the magnetic effect was also observed in micrometre-sized metal rings[6–8], demonstrating that electrons keep their phase coherence in such samples despite their diffusive motion. The search for the electrostatic contribution to the electron phase in these metal rings[9,10] was hindered by the high conductivity of metal, which makes it difficult to apply a well defined voltage difference across the ring. Here we report measurements of quantum interference of electrons in metal rings that are interrupted by two small tunnel junctions. In these systems, a well defined voltage difference between the two parts of the ring can be applied. Using these rings we simultaneously explore the influence of magnetic and electrostatic potentials on the Aharonov–Bohm quantum-interference effect, and we demonstrate that these two potentials play interchangeable roles.

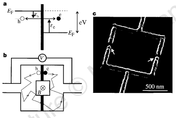


Figure 1 Quantum interference of electron–hole pair in mesoscopic ring. **a**, Energy diagram modelling a tunnelling process. At the moment of tunnelling, an electron (e) with energy ϵ_e is injected into the right part of the sample leaving a hole (h) with energy ϵ_h in the left part. **b**, Typical electron–hole trajectories which are sensitive to an Aharonov–Bohm flux. At the moment of a tunnel event an electron–hole pair is created. The hole and the electron diffuse separately to the other tunnel barrier where they recombine and interfere. The interference of the electron and hole wave is both sensitive to a magnetic field B piercing the ring (magnetic Aharonov–Bohm effect) and a voltage V across the ring (electrostatic Aharonov–Bohm effect). **c**, Scanning electron microscope image of the Aharonov–Bohm ring which is interrupted by two small tunnel junctions. The sample consists of a 0.9 μm × 0.9 μm square loop. The width of the arms of the loop is 60 nm. The loop is interrupted by two small tunnel junctions of 60 × 60 nm². The average distance L between the tunnel barriers is 1.6 μm. The mask of the loop was defined by electron beam lithography. The junctions were fabricated in three strips. First, a 25-nm-thick aluminium layer was deposited at an angle with respect to the oxidized Si substrate. Second, the aluminium was oxidized to form the insulating tunnel barriers. Finally, a second 40-nm-thick aluminium layer was deposited at another angle. The tunnel junctions are denoted by the arrows. By determining the overlap area of the different junctions, we conclude that the tunnelling conductances do not differ by more than 10%. This is confirmed by critical current measurements in the superconducting state, in which the sample operates as a SQUID. The tunnel conductance G_T of each junction is 55 ± 5 μS, which is about four orders of magnitude smaller than the conductance of the metallic part of the ring.

To determine the combined influence of electrostatic and magnetic potentials on the quantum interference of electron waves, we consider the model shown in Fig. 1a and b. A metallic ring is interrupted by two tunnel barriers denoted by the black regions in Fig. 1b. Because the resistance of these tunnel barriers is much larger than the resistance of the metallic part, a well defined potential V can be applied across the two halves of the ring. Electric transport occurs because electrons can tunnel from an occupied state in the left half to an empty state in the right half, as shown in Fig. 1a. This process is equivalent to the creation of an electron–hole pair during a tunnel event. After tunnelling, the electron and the hole start to diffuse in the right and left halves of the ring, respectively. Because the transport is diffusive, the electron and hole have a finite probability of recombining at the other tunnel barrier as shown in Fig. 1b. Only those trajectories in which the electron and hole trajectories form a closed loop circling the ring are sensitive to the magnetic field through the ring. At the moment of recombination, the phase difference $\Delta\phi_B$ between the electron and hole due to the magnetic field B is $2\pi eBS/h$ (where e is the electron charge, h is Planck's constant, and S is the area of the ring). This results in alternating constructive and destructive interference between the electron and hole wave as a function of B, with a period $h/(eS)$. This effect has analogies with the modulation of the Cooper pair current with period $h/(2eS)$ in a superconducting ring interrupted by tunnel junctions (a superconducting quantum interference device, SQUID)[11,12]. But in the case of the SQUID the effect results from collective Cooper pair tunnelling (the Josephson effect), whereas here the interference takes place at the single electron level. Because of the voltage difference, the electron and hole have different energies with respect to the Fermi level in both halves of our device. The sum of the energy of the hole ϵ_h and the electron ϵ_e

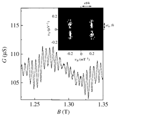


Figure 2 Conductance G versus magnetic field B at $T = 20$ mK and $V = 500$ μV. Aharonov–Bohm oscillations with a period $B_{AB} = 5$ mT are observed as a function of B. This period is in agreement with the magnetic field of 5.3 mT which is needed to add a flux quantum h/e to the average area $S = 0.75$ μm² enclosed by the ring. In the inset, the Fourier power spectrum is shown as a function of the magnetic and electric 'frequencies' ν_B and ν_V. The lighter the grey scale the larger the Fourier power. The magnetic and electric field modulation of G_{AB} manifests itself as peaks around ±0.2 mT⁻¹ and ±0.1 μV⁻¹, respectively.

equals eV (Fig. 1a). Owing to this energy difference, the electron–hole pair accumulates an electrostatic phase difference $\Delta\phi_V = 2\pi eVt/h$, where t is the time the electron and hole spend in their respective parts of the ring before they recombine. Because electron motion is diffusive there is not one unique time, but rather a distribution of times with an average value $t_0 = L^2/D$ where L is the distance along the ring between the two tunnel barriers and D is the diffusion coefficient. The electrostatic interference results in an alternating constructive and destructive interference as a function of V with a period $h/(et_0)$. Experimentally, one can explore both the magnetic and electrostatic quantum interference effect by measuring the Aharonov–Bohm flux-dependent part G_{AB} of the conductance. The magnetic field B and the bias voltage V have an equivalent effect on G_{AB}. Changing the magnetic field at fixed V leads to a periodically oscillating G_{AB} with a period which is solely defined by the geometry of the ring. Measuring the conductance as a function of V at fixed B results in a periodically oscillating G_{AB} with an average period which is determined by t_0. In other words, by exploring the voltage dependence of G_{AB}, t_0 is measured experimentally. In the ballistic regime related interference experiments[13] have been performed in which one of the arms of the Aharonov–Bohm ring was interrupted by a quantum dot. But the relevant interference processes in such systems, where the number of electrons in the ring is changing, differ significantly from the interference in the ring we consider here.

To investigate the combined role of magnetic and electrostatic potentials we designed the sample shown in Fig. 1c. The differential conductance G at a voltage V and magnetic field B was measured using a lock-in technique. Details of the measurement set-up are given in ref. 14. All measurements were performed at $B > 1$ T to drive the aluminium loop into the normal state. At those fields time-reversal symmetry is broken, and effects related to this symmetry can be neglected. In Fig. 2 the conductance G is plotted as a function of B at a bias voltage $V = 500$ μV and a temperature $T = 20$ mK. Clear periodic oscillations of the conductance are observed with a period of 5 mT, which is in good agreement with the predicted period for h/e oscillations. The relative amplitude of the Aharonov–Bohm oscillations at $V = 500$ μV is ~5%, which is considerably larger than the results obtained for uniform rings[6–10]. The magnetic field not only pierces the hole of the loop, but also penetrates the arms of the loop in contrast to the ideal geometry proposed by Aharonov and Bohm[1]. This gives rise to aperiodic conductance fluctuations[14], clearly visible in Fig. 2 as the slowly varying background on top of the Aharonov–Bohm oscillations. Because of the large difference in magnetic field scales, it is possible to filter out the aperiodic conductance fluctuations using Fourier analysis. In this way we can extract (from the conductance G) the Aharonov–Bohm conductance G_{AB}, which we would measure if the field was applied only inside the ring. The Fourier power spectrum of G_{AB} with respect to B and V is shown in Fig. 2 inset.

Figure 3 shows how the Aharonov–Bohm conductance G_{AB} evolves when the bias voltage is changed. The voltage increment between two successive traces is 0.48 μV. Trace a in Fig. 3 denotes a minimum of the Aharonov–Bohm conductance at $V = 519$ μV. When the voltage is decreased by 2.5 μV (trace b) the Aharonov–Bohm oscillations have almost vanished. A further decrease of the voltage by 2.5 μV (trace c) leads again to an increase of the oscillations. However, the minimum near $B = 1.032$ T for $V = 519$ μV (trace a) turned into a maximum at $V = 514$ μV (trace c). This observation unambiguously demonstrates the symmetry between the magnetic and the electrostatic Aharonov–Bohm effect. A maximum G_{AB} (constructive electron–hole interference) is turned into a minimum G_{AB} (destructive electron–hole interference) either by increasing the magnetic field by 2.5 mT or by increasing the voltage by 5 μV. This symmetry is also reflected in the Fourier power spectrum (Fig. 2 inset) by the peaks around ±0.2 mT⁻¹ and ±0.1 μV⁻¹.

To obtain a more quantitative analysis, we calculated the correlation function $C(\Delta B, \Delta V) = \langle G_{AB}(B, V)G_{AB}(B + \Delta B, V + \Delta V)\rangle$, where the angle brackets denote an ensemble average. The quantity is shown in Fig. 4. The squares in Fig. 4 denote the experimental $C(\Delta B, \Delta V = 0)$. Its oscillatory behaviour is again the manifestation of the magnetic Aharonov–Bohm effect (we denote the period by B_{AB}). The triangles in Fig. 4 denote the experimental cross-correlation

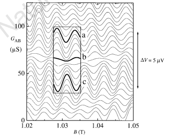


Figure 3 Aharonov–Bohm conductance as a function of magnetic field B at $T = 20$ mK for bias voltages separated by 0.48 μV. The traces have been offset by 5 μS with respect to the conductance-axis for clarity. The Aharonov–Bohm conductance G_{AB} has been extracted from G by rejecting the Fourier components of G for magnetic frequencies smaller than 0.1 mT⁻¹. The bold traces a–c in the rectangular box show an example of the combined manifestation of the magnetic and electrostatic Aharonov–Bohm effect.

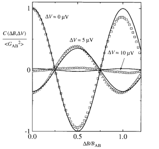
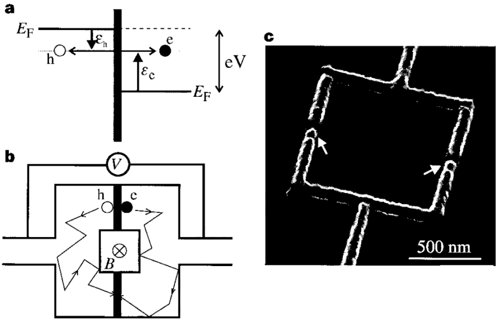


Figure 4 Normalized correlation function versus $\Delta B/B_{AB}$ at $T = 20$ mK. The correlation function $C(\Delta B, \Delta V) = \langle G_{AB}(B, V)G_{AB}(B + \Delta B, V + \Delta V)\rangle$ has been normalized to the variance $C(0, 0) = \langle G_{AB}^2\rangle$. Squares, triangles and circles correspond to $\Delta V = 0$, 5 and 10 μV, respectively. The correlation functions are calculated from a total data set consisting of 350 conductance traces as a function of B between 1.0 and 1.1 T or 2.0 and 2.1 T. These traces are measured at different voltages in the range 395–533 μV. The theoretical correlation function depends only on two parameters: t_0 and τ_ϕ, which is the time during which an electron–hole pair preserves its phase memory. A close agreement between the experimental results and theory (full lines) is found for $t_0 = 300$ ps and $\tau_\phi = 300$ ps. These values are consistent with previous experiments[14].

"Magneto-electric Aharonov-Bohm effect in metal rings," A. van Oudenaarden, et al., *Nature* **391**, 768-770 (1998).

Electrostatic A-B effect



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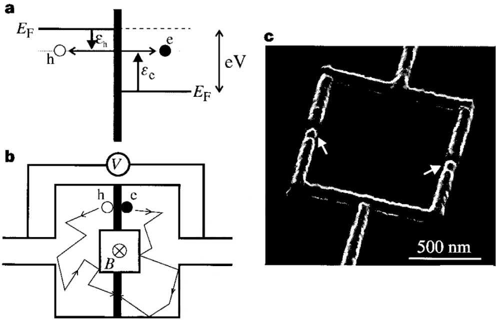

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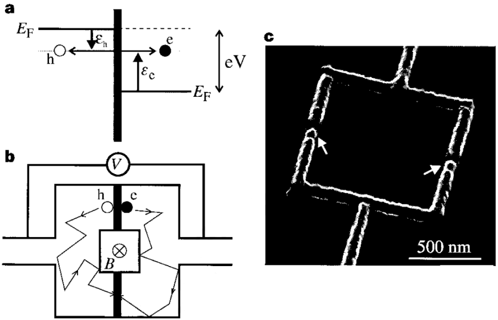

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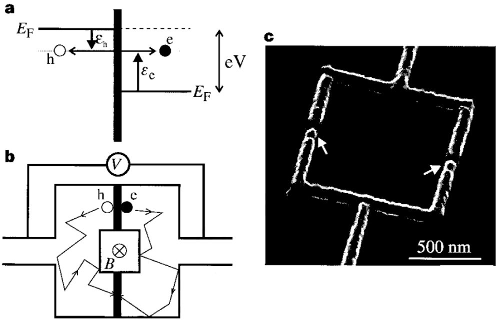

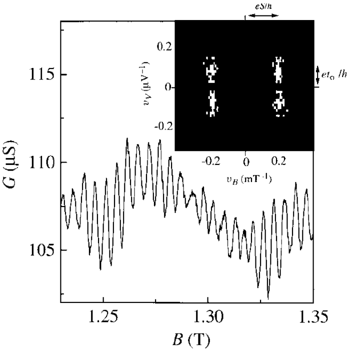

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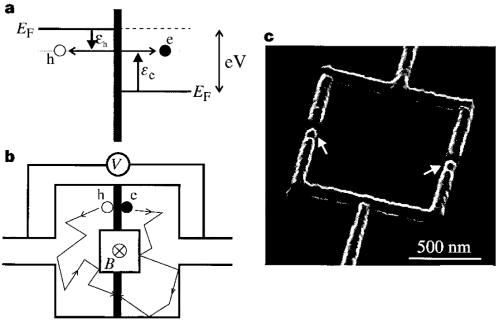

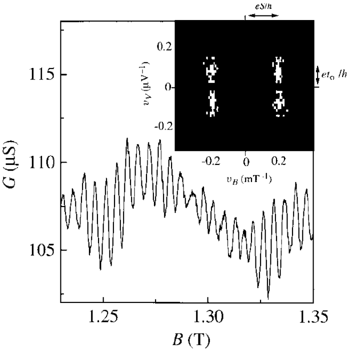

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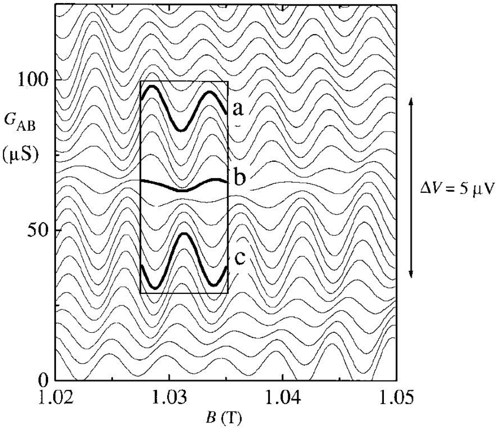

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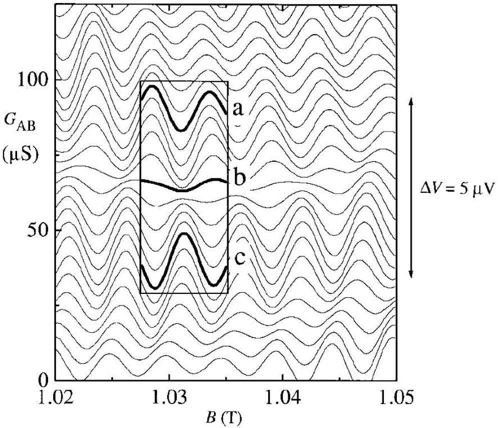

Aharonov-Bohm oscillations are seen also for constant magnetic field as applied potential is varied

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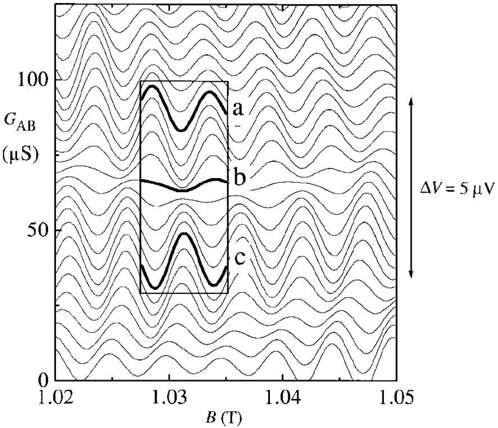

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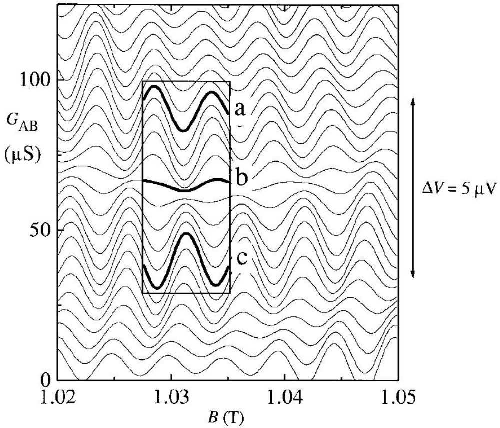

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Aharonov-Bohm effect continues to be an active area of research nearly 60 years after it was first proposed!

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"The Aharonov-Bohm effects: Variations on a subtle theme," H. Batelaan & A. Tonomura, *Physics Today* **62** (9), 38-43 (2009).

Adiabatic & semiclassical limits

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The adiabatic limit and the semiclassical limit

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Received 5 December 1983

Abstract. The evolution $|\Psi(t)\rangle$ of a system with slowly-varying Hamiltonian $\hat{H}(\Omega t)$ depends not only on the slowness parameter Ω but also on Planck's constant $\hbar$. For systems with two (or more) classically separated phase-space regions which are quantally connected by tunnelling, the curves of instantaneous energy levels display hyperbolic near-degeneracies rather than crossings. In such cases the limits $\Omega \to 0$ ($\hbar$ fixed and small) and $\hbar \to 0$ (Ω fixed and small) lead to opposite behaviour of $|\Psi(t)\rangle$. As an illustration, the uniform semiclassical adiabatic behaviour (Ω and $\hbar$ small, $\Omega/\hbar$ arbitrary) is calculated exactly for a double-well potential for which one well gets shallower as the other gets deeper.

1. Introduction

The adiabatic limit is the limit of slow change, and has given rise to two theorems, one for classical systems and one for quantal systems. In its simplest form, the classical adiabatic theorem (Arnol'd 1978) concerns integrable Hamiltonians $H(q_i, p_i; R_k(\Omega t))$ as $\Omega \to 0$, that is Hamiltonians depending on parameters R_k which vary slowly with time, as well as on N coordinates and momenta q_i and p_i, and whose orbits for fixed R_k are confined to N-tori in the $2N$-dimensional phase space. The theorem states that the action integrals I_j around the N irreducible cycles γ_j of the tori, defined as

$$I_j \equiv \frac{1}{2\pi}\oint_{\gamma_j} p_i \, dq_i, \tag{1}$$

are conserved in slow changes of the parameters R_k. The quantal adiabatic theorem (Messiah 1962) concerns evolution under the time-dependent Hamiltonian operator $\hat{H}(R_k(t)) \equiv H(\hat{q}_i, \hat{p}_i; R_k(\Omega t))$ and states that a system which starts at $t=0$ in an eigenstate of $\hat{H}(R_k(0))$ will remain for all t in the corresponding eigenstate of $\hat{H}(R_k(t))$, provided the $R_k(t)$ change slowly ($\Omega \to 0$) and the state is never degenerate.

It is natural to seek to connect these two theorems by means of the semiclassical limit, i.e. $\hbar \to 0$, and indeed such attempts played an important part in the development of quantum mechanics (Born 1960) by leading to the suggestion that (for integrable systems) the classical objects which correspond to quantum stationary states are phase-space tori. A strong form of this connection has been asserted by Hwang and Pechukas (1977), who claimed that the asymptotic limits $\hbar \to 0$ and $\Omega \to 0$ are equivalent. Their argument is based on scaling: in the Schrödinger equation

$$i\hbar \, \partial|\Psi\rangle/\partial t = \hat{H}(R_k(\Omega t))|\Psi\rangle, \tag{2}$$

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Ω can be eliminated from $\hat{H}$ by defining $\Omega t = \tau$, leading to the replacement of $\hbar$ by $\Omega\hbar$ on the left-hand side. This argument is ill-founded because the notation $\hat{H}$ conceals an $\hbar$-dependence (whose explicit form is different for different representations) which persists after t-scaling, leaving an equation depending on $\hbar$ as well as $\Omega\hbar$.

My purpose here is to point out that there is even a class of systems for which the semiclassical limit and the adiabatic limit flatly contradict each other. These systems involve pairs of quantum states associated with classical trajectories in separate regions of classical phase space, which are connected quantally by *tunnelling*. A simple model for such systems is set up in § 2 and solved in § 3. Some generalisations, and also possible implications for the difficult problem of semiclassical quantisation of non-integrable systems are discussed in § 4.

2. The changing double-well potential

Consider a particle of mass m moving in one dimension with energy E in the time-dependent potential $V(q, t)$, illustrated in figure 1, whose left-hand well (called L) gets shallower and whose right-hand well (R) gets deeper as shape parameters $R_k(t)$ change slowly. Only energies E less than the energy of the barrier top will be considered. Thus there are two distinct classical motions with each E, and two actions I_L and I_R, which may be considered to be the parameters $R_k(t)$ and which are given (cf (1)) by

$$\begin{aligned} I_L(E,t) &= \frac{1}{\pi}\int_{q_{L-}}^{q_{L+}} dq\,[2m(E - V(q,t))]^{1/2}, \\ I_R(E,t) &= \frac{1}{\pi}\int_{q_{R-}}^{q_{R+}} dq\,[2m(E - V(q,t))]^{1/2}, \end{aligned} \tag{3}$$

where the limits of integration are the classical turning points (figure 1).

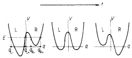

Figure 1. Changing double-well potential.

In the simplest semiclassical approximation, each well supports separate families of localised quantum stationary states $|\phi_L^m\rangle$ and $|\phi_R^n\rangle$ with quantum numbers n and m; the exponential tails leaking out of each well may be neglected. The energies $E_{L,m}(t)$, $E_{R,n}(t)$ of the states are given by the Bohr–Sommerfeld rule

$$\left.\begin{aligned} I_L(E_{L,m}(t), t) &= (m+\tfrac{1}{2})\hbar \\ I_R(E_{R,n}(t), t) &= (n+\tfrac{1}{2})\hbar \end{aligned}\right\}. \tag{4}$$

Successive levels in each family are separated by $\hbar\omega_L$ and $\hbar\omega_R$, where $\omega = (\partial I/\partial E)^{-1}$ is the frequency of classical motion in each well. As I_L and I_R change, the energies

"The adiabatic limit and the semiclassical limit," M.V. Berry, *J. Phys. A: Math. Gen.* **17**, 1225-1233 (1984).

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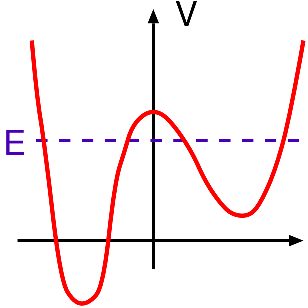

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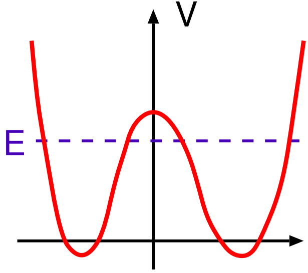

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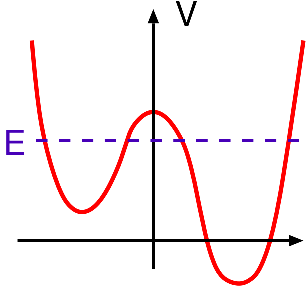

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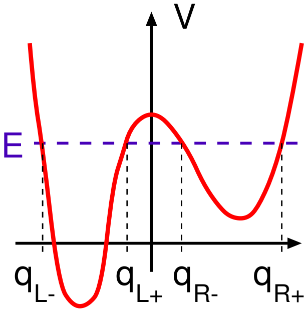

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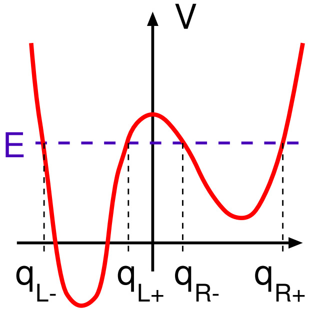

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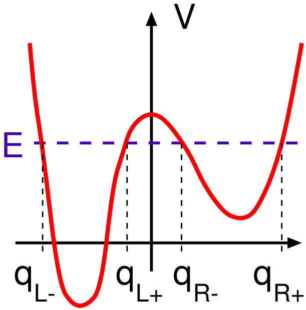

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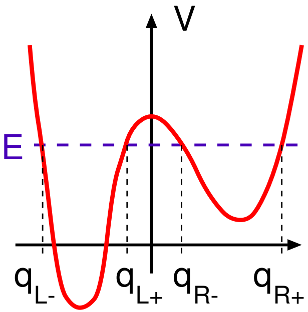

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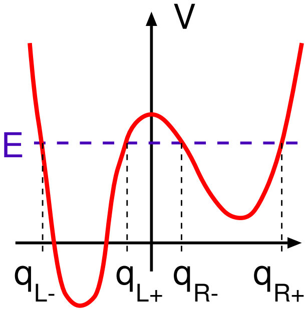

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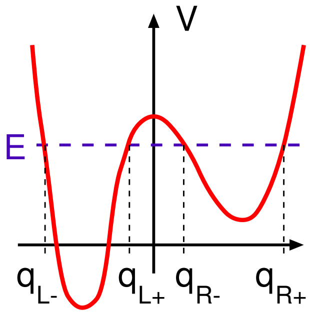

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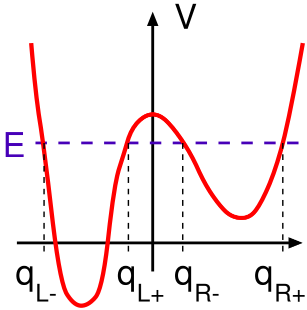

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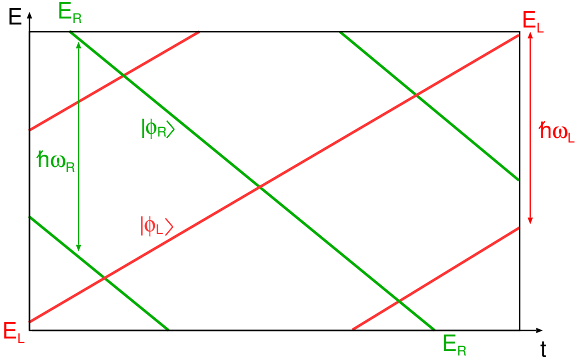

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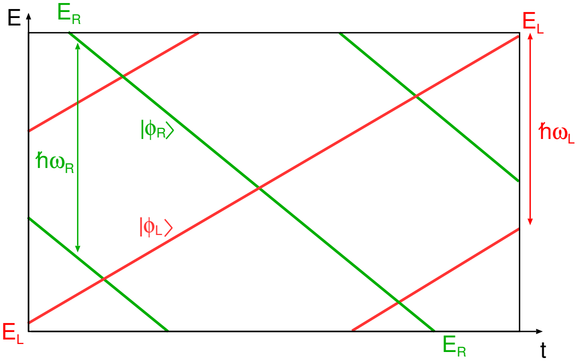

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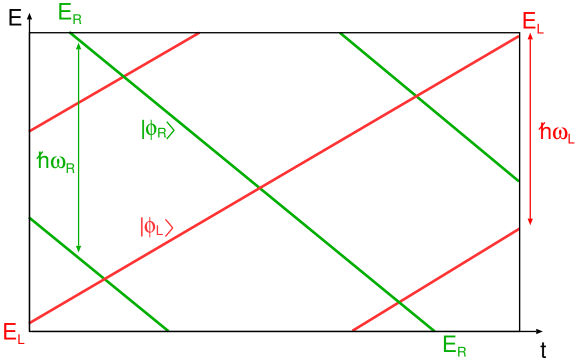

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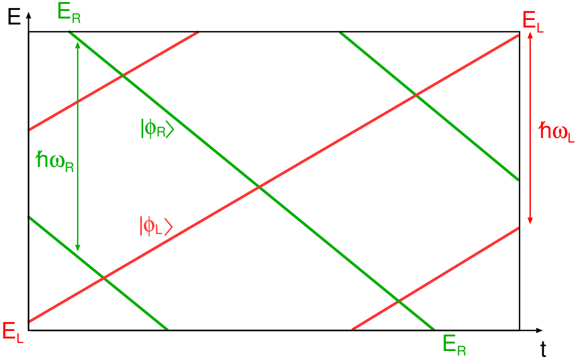

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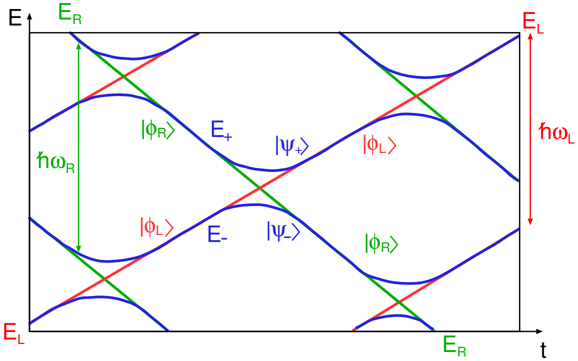

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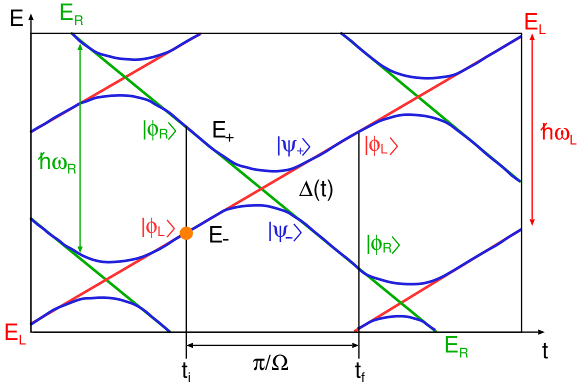

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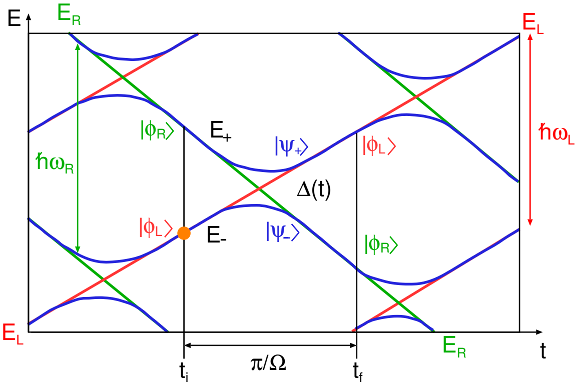

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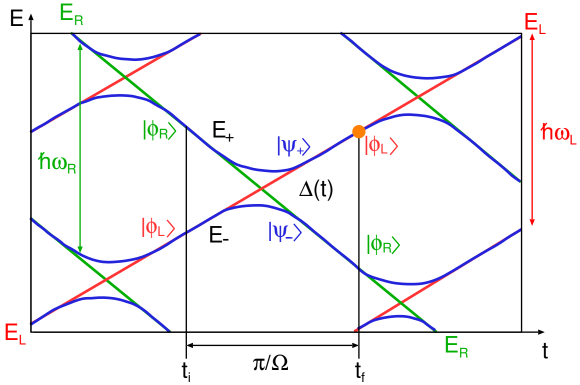

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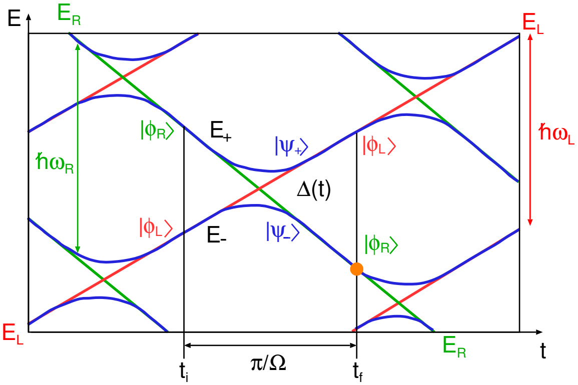

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according to the quantum adiabatic approximation, however, the system will remain in $|\psi_-\rangle$ throughout and thus find itself in $|\phi_R\rangle$ at t_f

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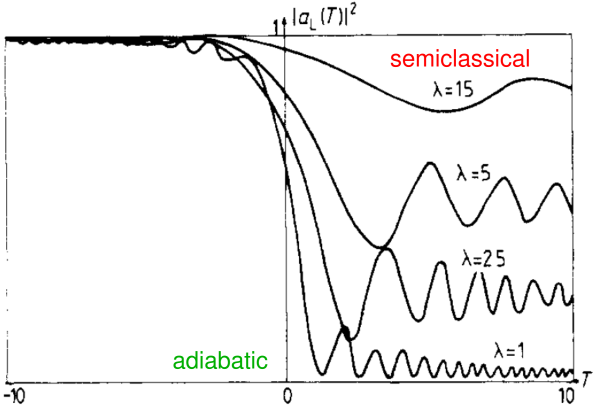

Quantum paradoxes and other fun stuff

"About your cat, Mr. Schrödinger – I have good news and bad news . . ."

Einstein Podolsky Rosen paradox

of lanthanum is 7/2, hence the nuclear magnetic moment as determined by this analysis is 2.5 nuclear magnetons. This is in fair agreement with the value 2.8 nuclear magnetons determined from La III hyperfine structures by the writer and N. S. Grace.[9]

[9] M. F. Crawford and N. S. Grace, Phys. Rev. **47**, 536 (1935).

This investigation was carried out under the supervision of Professor G. Breit, and I wish to thank him for the invaluable advice and assistance so freely given. I also take this opportunity to acknowledge the award of a Fellowship by the Royal Society of Canada, and to thank the University of Wisconsin and the Department of Physics for the privilege of working here.

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Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

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In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty* (*i.e.*, *with probability equal to unity*) *the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity*. It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one

such way, whenever the conditions set down in it occur. Regarded not as a necessary, but merely as a sufficient, condition of reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality.

To illustrate the ideas involved let us consider the quantum-mechanical description of the behavior of a particle having a single degree of freedom. The fundamental concept of the theory is the concept of *state*, which is supposed to be completely characterized by the wave function ψ, which is a function of the variables chosen to describe the particle's behavior. Corresponding to each physically observable quantity A there is an operator, which may be designated by the same letter.

If ψ is an eigenfunction of the operator A, that is, if

$$\psi' \equiv A\psi = a\psi, \tag{1}$$

where a is a number, then the physical quantity A has with certainty the value a whenever the particle is in the state given by ψ. In accordance with our criterion of reality, for a particle in the state given by ψ for which Eq. (1) holds, there is an element of physical reality corresponding to the physical quantity A. Let, for example,

$$\psi = e^{(2\pi i/h)p_0 x}, \tag{2}$$

where h is Planck's constant, p_0 is some constant number, and x the independent variable. Since the operator corresponding to the momentum of the particle is

$$p = (h/2\pi i)\partial/\partial x, \tag{3}$$

we obtain

$$\psi' = p\psi = (h/2\pi i)\partial\psi/\partial x = p_0\psi. \tag{4}$$

Thus, in the state given by Eq. (2), the momentum has certainly the value p_0. It thus has meaning to say that the momentum of the particle in the state given by Eq. (2) is real.

On the other hand if Eq. (1) does not hold, we can no longer speak of the physical quantity A having a particular value. This is the case, for example, with the coordinate of the particle. The operator corresponding to it, say q, is the operator of multiplication by the independent variable. Thus,

$$q\psi = x\psi \neq a\psi. \tag{5}$$

In accordance with quantum mechanics we can only say that the relative probability that a measurement of the coordinate will give a result lying between a and b is

$$P(a,b) = \int_a^b \bar{\psi}\psi dx = \int_a^b dx = b - a. \tag{6}$$

Since this probability is independent of a, but depends only upon the difference $b-a$, we see that all values of the coordinate are equally probable.

A definite value of the coordinate, for a particle in the state given by Eq. (2), is thus not predictable, but may be obtained only by a direct measurement. Such a measurement however disturbs the particle and thus alters its state. After the coordinate is determined, the particle will no longer be in the state given by Eq. (2). The usual conclusion from this in quantum mechanics is that *when the momentum of a particle is known, its coordinate has no physical reality.*

More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if $AB \neq BA$, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first.

From this follows that either (1) *the quantum-mechanical description of reality given by the wave function is not complete* or (2) *when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality*. For if both of them had simultaneous reality—and thus definite values—these values would enter into the complete description, according to the condition of completeness. If then the wave function provided such a complete description of reality, it would contain these values; these would then be predictable. This not being the case, we are left with the alternatives stated.

In quantum mechanics it is usually assumed that the wave function *does* contain a complete description of the physical reality of the system in the state to which it corresponds. At first

"Can quantum-mechanical description of physical reality be considered complete?," A. Einstein, B. Podolsky, and N. Rosen, *Physical Review* **47**, 777-779 (1935).

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Thus, as a consequence of two different measurements made on System I, System II can be left in states with two different wave functions, even when it is far away from, and not interacting with System I

Einstein Podolsky Rosen paradox

If instead of operator A, we choose to expand the overall wavefunction in terms of the eigenfunctions of B, then we have

$$b_1, b_2, b_3, \dots$$

$$v_1(x_1), v_2(x_1), v_3(x_1), \dots$$

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Thus one can assign two different wave functions, $\psi_k(x_2)$ and $\varphi_r(x_2)$, to the same reality (System II after interaction with System I)

Einstein Podolsky Rosen paradox

Suppose that $\psi_k(x_2)$ and $\varphi_r(x_2)$ are eigenfunctions of two non-commuting operators, P and Q with eigenvalues p_k and q_r

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By measuring either A or B on System I, we are able to predict with certainty, and without disturbing System II, either the value of P (p_k) or Q (q_r)

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If this "realist" version of quantum mechanics is correct, the "complete" description of reality must include some local hidden variable(s) which specify the state of the system completely