Today's Outline - February 20, 2013

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Reading Assignment: Chapter 8.3

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Note: deal with turning points later...

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Starting with the Schrödinger equation and $E > V(x)$

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taking the derivatives and substituting into the Schrödinger equation

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$$\psi(x) \simeq \frac{C}{\sqrt{\phi'}}e^{\pm\frac{i}{\hbar}\int p(x)\,dx}$$

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When $f(x) \equiv 0$ we recover the energy of the infinite square well since $pa = \sqrt{2mE}a = n\pi\hbar$