The 3D radial equation

The 3D radial equation

- Recasting the radial equation

The 3D radial equation

- Recasting the radial equation
- Particle in a bubble solution

Radial solution

Returning to the radial equation, where the potential is included

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right)$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \left[r\frac{du}{dr} - u\right]\frac{1}{r^2}$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \left[r\frac{du}{dr} - u\right]\frac{1}{r^2}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \left[r\frac{du}{dr} - u\right]\frac{1}{r^2}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left[r\frac{du}{dr} - u\right]$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \left[r\frac{du}{dr} - u\right]\frac{1}{r^2}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left[r\frac{du}{dr} - u\right] = \frac{du}{dr} + r\frac{d^2u}{dr^2} - \frac{du}{dr}$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \left[r\frac{du}{dr} - u\right]\frac{1}{r^2}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left[r\frac{du}{dr} - u\right] = \frac{du}{dr} + r\frac{d^2u}{dr^2} - \frac{du}{dr} = r\frac{d^2u}{dr^2}$$

Radial solution

Returning to the radial equation, where the potential is included

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

A change of variables to $u(r) \equiv rR(r)$ simplifies the equation

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \left[r\frac{du}{dr} - u\right]\frac{1}{r^2}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left[r\frac{du}{dr} - u\right] = \frac{du}{dr} + r\frac{d^2u}{dr^2} - \frac{du}{dr} = r\frac{d^2u}{dr^2}$$

$$r\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]\frac{u}{r} = l(l+1)\frac{u}{r}$$

Equation for u(r)

We can now obtain a differential equation for u(r)

$$r\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]\frac{u}{r} = l(l+1)\frac{u}{r}$$

Equation for u(r)

We can now obtain a differential equation for u(r)

$$r\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]\frac{u}{r} = l(l+1)\frac{u}{r}$$

multiplying by r,

$$r^2\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]u = l(l+1)u$$

Equation for u(r)

We can now obtain a differential equation for u(r)

$$r\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}[V(r) - E]\frac{u}{r} = l(l+1)\frac{u}{r}$$

multiplying by r, rearranging,

$$r^2\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}[V(r) - E]u = l(l+1)u$$

$$r^2\frac{d^2u}{dr^2} - \left[\frac{2mr^2}{\hbar^2}V(r)u + l(l+1)u\right] = -\frac{2mr^2}{\hbar^2}Eu$$

Equation for u(r)

We can now obtain a differential equation for u(r)

$$r\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]\frac{u}{r} = l(l+1)\frac{u}{r}$$

multiplying by r, rearranging, and multiplying by $-\hbar^2/2mr^2$

$$r^2\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]u = l(l+1)u$$

$$r^2\frac{d^2u}{dr^2} - \left[\frac{2mr^2}{\hbar^2}V(r)u + l(l+1)u\right] = -\frac{2mr^2}{\hbar^2}Eu$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

Equation for u(r)

We can now obtain a differential equation for u(r)

$$r\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}[V(r) - E]\frac{u}{r} = l(l+1)\frac{u}{r}$$

multiplying by r, rearranging, and multiplying by $-\hbar^2/2mr^2$

$$r^2\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}[V(r) - E]\,u = l(l+1)u$$

$$r^2\frac{d^2u}{dr^2} - \left[\frac{2mr^2}{\hbar^2}V(r)u + l(l+1)u\right] = -\frac{2mr^2}{\hbar^2}Eu$$

This looks like a time-independent Schrödinger equation with an effective potential $V_{eff}(r)$ whose solution is normalized as $\int_0^\infty |u(r)|^2\,dr = 1$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

Example 4.1

Consider the potential for the infinite spherical well

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

inside the well the wave equation is

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

inside the well the wave equation is

$$Eu = -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u$$

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

inside the well the wave equation is

$$Eu = -\frac{\hbar^2}{2m}\frac{d^2 u}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u$$

dividing by $\hbar^2/2m$ and rearranging

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

inside the well the wave equation is

$$Eu = -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u$$

dividing by $\hbar^2/2m$ and rearranging

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - \frac{2mE}{\hbar^2}\right]u$$

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

inside the well the wave equation is

$$Eu = -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u$$

dividing by $\hbar^2/2m$ and rearranging

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - \frac{2mE}{\hbar^2}\right]u$$

substituting $k = \sqrt{2mE}/\hbar$

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

inside the well the wave equation is

$$Eu = -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u$$

dividing by $\hbar^2/2m$ and rearranging

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - \frac{2mE}{\hbar^2}\right]u$$

substituting $k = \sqrt{2mE}/\hbar$

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - k^2\right]u$$

Example 4.1

Consider the potential for the infinite spherical well

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty, & r \geq a \end{cases}$$

inside the well the wave equation is

$$Eu = -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u$$

dividing by $\hbar^2/2m$ and rearranging

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - \frac{2mE}{\hbar^2}\right]u$$

substituting $k = \sqrt{2mE}/\hbar$

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - k^2\right]u$$

We must solve this for each value of l separately. The $l = 0$ case is the simplest

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2u$$

this has the usual solution

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + B\cos(kr)$$

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + B\cos(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + \cancel{B}\cos(kr)$$
$$= A\sin(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + \cancel{B}\cos(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

$$= A\sin(kr)$$

applying the boundary condition

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + \cancel{B}\cos(kr)$$
$$= A\sin(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

applying the boundary condition

$$u(a) = A\sin(ka) = 0$$

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + \cancel{B}\cos(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

$$= A\sin(kr)$$

applying the boundary condition

$$u(a) = A\sin(ka) = 0$$

thus, $ka = n\pi$ where n is an integer and the energies for the $l = 0$ case are

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + \cancel{B}\cos(kr)$$
$$= A\sin(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

applying the boundary condition

$$u(a) = A\sin(ka) = 0$$

thus, $ka = n\pi$ where n is an integer and the energies for the $l = 0$ case are

$$E_{n0} = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots$$

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + \cancel{B}\cos(kr)$$
$$= A\sin(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

applying the boundary condition

$$u(a) = A\sin(ka) = 0$$

thus, $ka = n\pi$ where n is an integer and the energies for the $l = 0$ case are

$$E_{n0} = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots$$

Normalizing ($A = \sqrt{2/a}$) and adding the angular part of the wave function $Y_0^0 = 1/\sqrt{4\pi}$

Example 4.1 (cont.)

For $l = 0$, we need to impose the boundary condition $u(a) = 0$.

$$\frac{d^2u}{dr^2} = -k^2 u$$

this has the usual solution

$$u(r) = A\sin(kr) + \cancel{B}\cos(kr)$$
$$= A\sin(kr)$$

since $R(r) = u/r$, the $\cos(kr)$ solution would be infinite at the origin and thus $B = 0$

applying the boundary condition

$$u(a) = A\sin(ka) = 0$$

thus, $ka = n\pi$ where n is an integer and the energies for the $l = 0$ case are

$$E_{n0} = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots$$

Normalizing ($A = \sqrt{2/a}$) and adding the angular part of the wave function $Y_0^0 = 1/\sqrt{4\pi}$

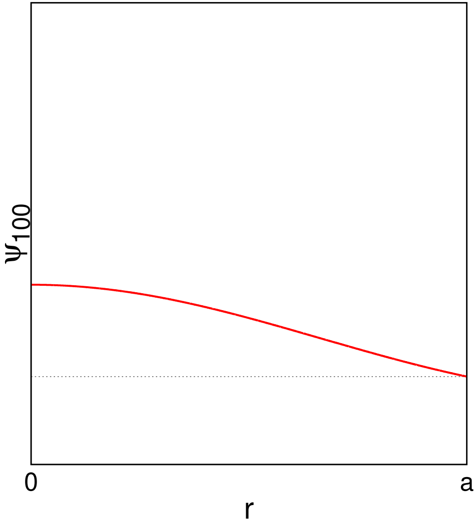
$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$$

Solutions for $l = 0$

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$$

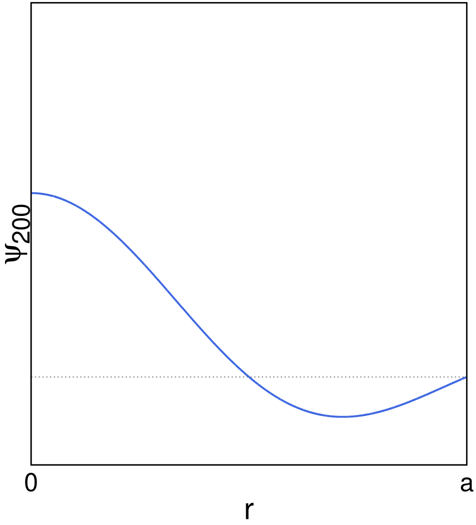
Solutions for $l = 0$

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$$



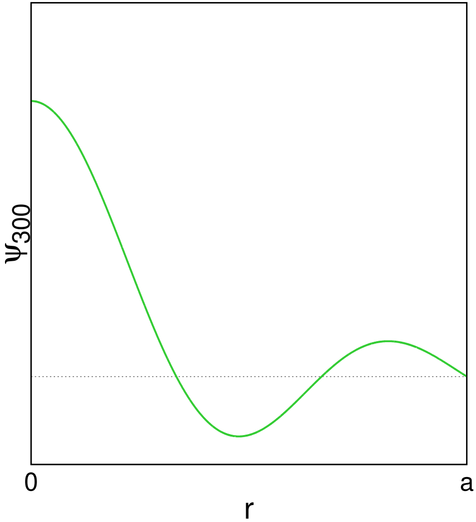
Solutions for $l = 0$

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$$



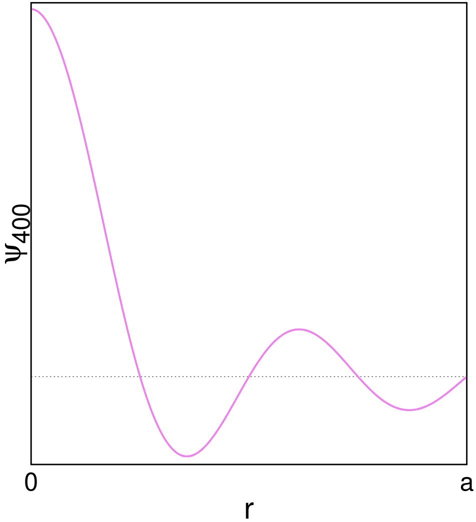
Solutions for $l = 0$

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$$



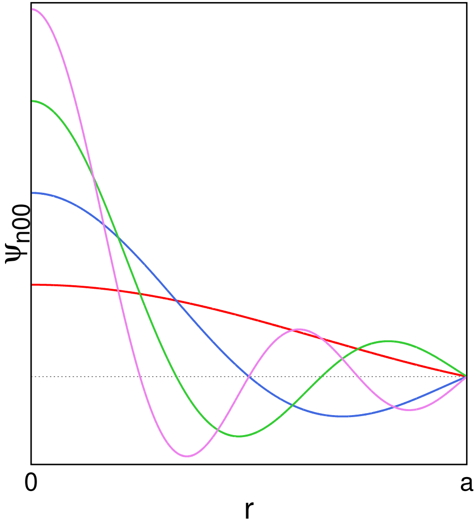
Solutions for $l = 0$

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$$



Solutions for $l = 0$

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$$



Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

they involve Bessel functions; the general solution being

Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

they involve Bessel functions; the general solution being

$$u(r) = A_l \, r \, j_l(kr) + B_l \, r \, n_l(kr)$$

Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

they involve Bessel functions; the general solution being

$$u(r) = A_l \, r \, j_l(kr) + B_l \, r \, n_l(kr)$$

$j_l(x)$ are spherical Bessel functions of order l

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x}$$

Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

they involve Bessel functions; the general solution being

$$u(r) = A_l \, r \, j_l(kr) + B_l \, r \, n_l(kr)$$

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x}$$

$$n_l(x) = -(-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\cos x}{x}$$

$j_l(x)$ are spherical Bessel functions of order l and $n_l(x)$ are spherical Neumann functions of order l.

Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

they involve Bessel functions; the general solution being

$$u(r) = A_l \, r \, j_l(kr) + B_l \, r \, n_l(kr)$$

$j_l(x)$ are spherical Bessel functions of order l and $n_l(x)$ are spherical Neumann functions of order l.

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x}$$

$$n_l(x) = -(-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\cos x}{x}$$

while spherical Bessel functions are finite at the origin, spherical Neumann functions are infinite and thus we again set $B_l = 0$

Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

they involve Bessel functions; the general solution being

$$u(r) = A_l \, r \, j_l(kr) + B_l \, r \, n_l(kr)$$

$j_l(x)$ are spherical Bessel functions of order l and $n_l(x)$ are spherical Neumann functions of order l.

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x}$$

$$n_l(x) = -(-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\cos x}{x}$$

while spherical Bessel functions are finite at the origin, spherical Neumann functions are infinite and thus we again set $B_l = 0$

$$R(r) = A_l \, j_l(kr)$$

Example 4.1 (cont.)

What about the solutions for $l \neq 0$?

they involve Bessel functions; the general solution being

$$u(r) = A_l \, r \, j_l(kr) + B_l \, r \, n_l(kr)$$

$j_l(x)$ are spherical Bessel functions of order l and $n_l(x)$ are spherical Neumann functions of order l.

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x}$$

$$n_l(x) = -(-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\cos x}{x}$$

while spherical Bessel functions are finite at the origin, spherical Neumann functions are infinite and thus we again set $B_l = 0$

$$R(r) = A_l \, j_l(kr)$$

we still must apply the boundary condition that $j_l(ka) = 0$ but this is a bit more complex than for the $l = 0$ case

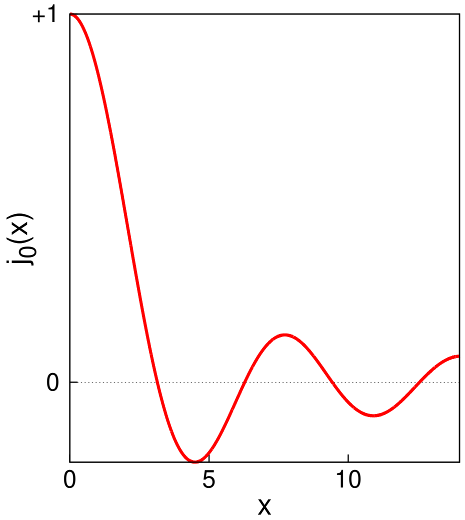
Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x}$$

Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

$$j_0(x) = \frac{\sin x}{x}$$

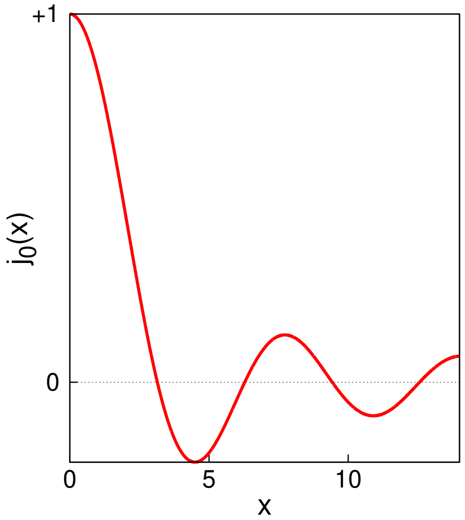


Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = (-x)\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)$$

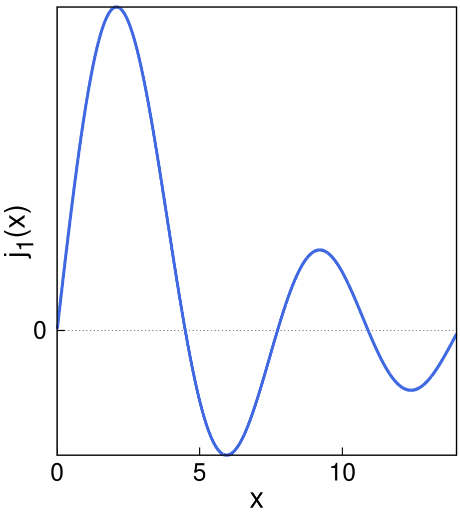


Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = (-x)\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$



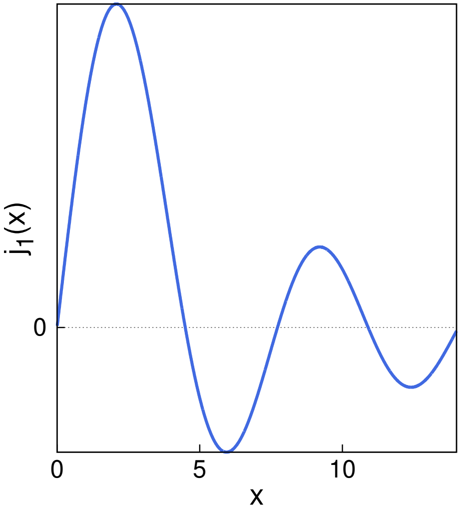
Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = (-x)\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = (-x)^2 \left(\frac{1}{x}\frac{d}{dx}\right) \frac{x\cos x - \sin x}{x^3}$$



Spherical Bessel functions

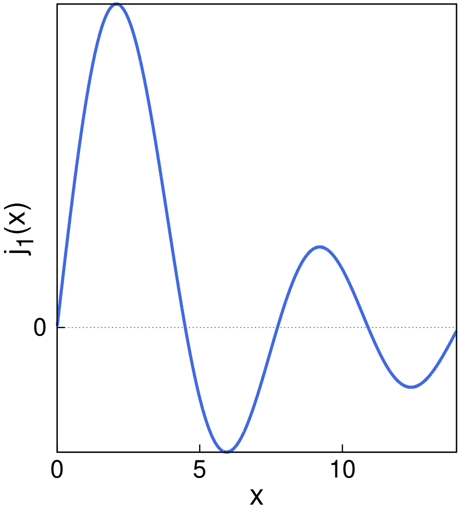
$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = (-x)\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = (-x)^2 \left(\frac{1}{x}\frac{d}{dx}\right) \frac{x\cos x - \sin x}{x^3}$$

$$= x\left(-\frac{x\sin x}{x^3} - 3\frac{x\cos x - \sin x}{x^4}\right)$$



Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

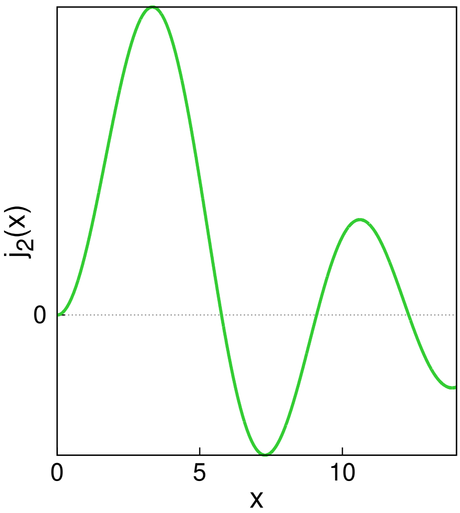
$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = (-x)\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = (-x)^2 \left(\frac{1}{x}\frac{d}{dx}\right) \frac{x\cos x - \sin x}{x^3}$$

$$= x\left(-\frac{x \sin x}{x^3} - 3\frac{x\cos x - \sin x}{x^4}\right)$$

$$= \left(\frac{3\sin x - 3x\cos x - x^2 \sin x}{x^3}\right)$$



Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

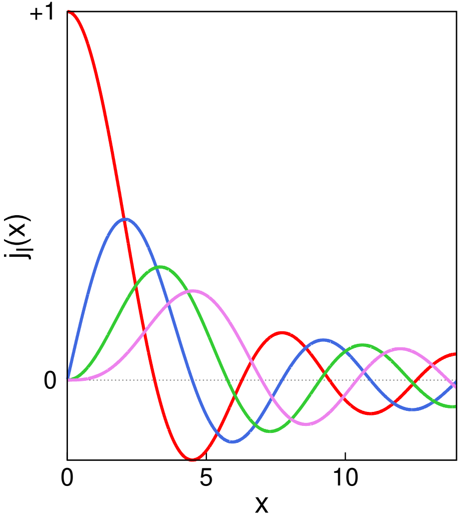
$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = (-x)\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = (-x)^2 \left(\frac{1}{x}\frac{d}{dx}\right) \frac{x\cos x - \sin x}{x^3}$$

$$= x\left(-\frac{x\sin x}{x^3} - 3\frac{x\cos x - \sin x}{x^4}\right)$$

$$= \left(\frac{3\sin x - 3x\cos x - x^2\sin x}{x^3}\right)$$



Clearly the roots are not at nice, simple, locations!

Solutions for all l

$$j_l(ka) = 0$$

Solutions for all l

$$j_l(ka) = 0$$

$$k = \frac{1}{a}\beta_{nl}, \qquad \beta_{nl} \text{ are the roots}$$

Solutions for all l

$$j_l(ka) = 0$$

$$k = \frac{1}{a}\beta_{nl}, \qquad \beta_{nl} \text{ are the roots}$$

$$E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$$

Solutions for all l

$$j_l(ka) = 0$$

$$k = \frac{1}{a}\beta_{nl}, \qquad \beta_{nl} \text{ are the roots}$$

$$E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$$

	β_{n1}	β_{n2}	β_{n3}
j_0	π	2π	3π
j_1	4.493	7.726	10.904
j_2	5.762	9.906	12.325
j_3	6.988	10.420	13.698

Solutions for all l

$$j_l(ka) = 0$$

$$k = \frac{1}{a}\beta_{nl}, \qquad \beta_{nl} \text{ are the roots}$$

$$E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$$

The wavefunctions are thus:

	β_{n1}	β_{n2}	β_{n3}
j_0	π	2π	3π
j_1	4.493	7.726	10.904
j_2	5.762	9.906	12.325
j_3	6.988	10.420	13.698

Solutions for all l

$$j_l(ka) = 0$$
$$k = \frac{1}{a}\beta_{nl}, \qquad \beta_{nl} \text{ are the roots}$$
$$E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$$

	β_{n1}	β_{n2}	β_{n3}
j_0	π	2π	3π
j_1	4.493	7.726	10.904
j_2	5.762	9.906	12.325
j_3	6.988	10.420	13.698

The wavefunctions are thus:

$$\psi_{nlm}(r,\theta,\phi) = A_{nl}\, j_l\left(\frac{\beta_{nl} r}{a}\right) Y_l^m(\theta,\phi)$$

Solutions for all l

$$j_l(ka) = 0$$

$$k = \frac{1}{a}\beta_{nl}, \qquad \beta_{nl} \text{ are the roots}$$

$$E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$$

	β_{n1}	β_{n2}	β_{n3}
j_0	π	2π	3π
j_1	4.493	7.726	10.904
j_2	5.762	9.906	12.325
j_3	6.988	10.420	13.698

The wavefunctions are thus:

$$\psi_{nlm}(r,\theta,\phi) = A_{nl}\, j_l\left(\frac{\beta_{nl} r}{a}\right) Y_l^m(\theta,\phi)$$

these are $(2l+1)$-fold degenerate states, that is, the energy does not depend on the quantum numbers which give rise to the degeneracy

Hydrogen atom: Part 1

Hydrogen atom: Part 1

- Hydrogen atom potential

Hydrogen atom: Part 1

- Hydrogen atom potential
- Asymptotic solution

Hydrogen atom: Part 1

- Hydrogen atom potential
- Asymptotic solution
- Differential equation for polynomial

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\quad V(r) \quad + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = Eu$$

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\quad V(r) \quad + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = Eu$$

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

dividing by E and rearranging

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

dividing by E and rearranging

$$-\frac{\hbar^2}{2mE}\frac{d^2u}{dr^2} = \left[1 + \frac{e^2}{4\pi\epsilon_0 E}\frac{1}{r} - \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right]u$$

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

dividing by E and rearranging

$$-\frac{\hbar^2}{2mE}\frac{d^2u}{dr^2} = \left[1 + \frac{e^2}{4\pi\epsilon_0 E}\frac{1}{r} - \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right]u$$

rewriting it with common terms

$$-\frac{\hbar^2}{2mE}\frac{d^2u}{dr^2} = \left[1 + \frac{me^2}{2\pi\epsilon_0\hbar^2}\frac{\hbar^2}{2mE}\frac{1}{r} - \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right]u$$

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

dividing by E and rearranging

$$-\frac{\hbar^2}{2mE}\frac{d^2u}{dr^2} = \left[1 + \frac{e^2}{4\pi\epsilon_0 E}\frac{1}{r} - \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right]u$$

rewriting it with common terms

$$-\frac{\hbar^2}{2mE}\frac{d^2u}{dr^2} = \left[1 + \frac{me^2}{2\pi\epsilon_0\hbar^2}\frac{\hbar^2}{2mE}\frac{1}{r} - \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right]u$$

Hydrogen atom

The potential of the hydrogen atom is simply the Coulomb potential, which is spherically symmetric

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

where we assume that the nucleus (proton) is stationary because it is much more massive than the electron

we can substitute this potential into the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

dividing by E and rearranging

$$-\frac{\hbar^2}{2mE}\frac{d^2u}{dr^2} = \left[1 + \frac{e^2}{4\pi\epsilon_0 E}\frac{1}{r} - \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right]u$$

rewriting it with common terms

$$-\frac{\hbar^2}{2mE}\frac{d^2u}{dr^2} = \left[1 + \frac{me^2}{2\pi\epsilon_0\hbar^2}\frac{\hbar^2}{2mE}\frac{1}{r} - \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right]u$$

initially we are only interested in bound states with $E < 0$ and so we can make the usual substitution $\kappa = \sqrt{-2mE}/\hbar$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

just as with the harmonic oscillator, we start with the asymptotic solution and then generalize

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

As $\rho \to \infty$, the constant term dominates

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

just as with the harmonic oscillator, we start with the asymptotic solution and then generalize

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

As $\rho \to \infty$, the constant term dominates

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

just as with the harmonic oscillator, we start with the asymptotic solution and then generalize

$$\frac{d^2u}{d\rho^2} \approx u$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

As $\rho \to \infty$, the constant term dominates and the solution is of the form

just as with the harmonic oscillator, we start with the asymptotic solution and then generalize

$$\frac{d^2u}{d\rho^2} \approx u$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

just as with the harmonic oscillator, we start with the asymptotic solution and then generalize

As $\rho \to \infty$, the constant term dominates and the solution is of the form

$$\frac{d^2u}{d\rho^2} \approx u$$

$$u(\rho)|_{\rho\to\infty} = Ae^{-\rho} + Be^{\rho}$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

As $\rho \to \infty$, the constant term dominates

and the solution is of the form

but the second term is unbounded in the limit of $\rho \to \infty$, thus $B = 0$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

just as with the harmonic oscillator, we start with the asymptotic solution and then generalize

$$\frac{d^2u}{d\rho^2} \approx u$$

$$u(\rho)|_{\rho\to\infty} = Ae^{-\rho} + \cancel{Be^{\rho}}$$

Hydrogen atom (cont.)

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r)^2}\right]u$$

if we substitute

$$\rho \equiv \kappa r, \qquad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$$

just as with the harmonic oscillator, we start with the asymptotic solution and then generalize

As $\rho \to \infty$, the constant term dominates

$$\frac{d^2u}{d\rho^2} \approx u$$

and the solution is of the form

$$u(\rho)|_{\rho\to\infty} = Ae^{-\rho} + \cancel{Be^{\rho}}$$

but the second term is unbounded in the limit of $\rho \to \infty$, thus $B = 0$

however, in the limit of $\rho \to 0$, the centrifugal term is dominant

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

$$\frac{du}{d\rho} = (l+1)C\rho^l - l\frac{D}{\rho^{(l+1)}}$$

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

$$\frac{du}{d\rho} = (l+1)C\rho^l - l\frac{D}{\rho^{(l+1)}}$$

$$\frac{d^2u}{d\rho^2} = l(l+1)C\rho^{l-1} + l(l+1)\frac{D}{\rho^{(l+2)}}$$

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

$$\frac{du}{d\rho} = (l+1)C\rho^l - l\frac{D}{\rho^{(l+1)}}$$

$$\frac{d^2u}{d\rho^2} = l(l+1)C\rho^{l-1} + l(l+1)\frac{D}{\rho^{(l+2)}}$$

but the second term blows up as $\rho \to 0$, so $D = 0$ and

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

$$\frac{du}{d\rho} = (l+1)C\rho^l - l\frac{D}{\rho^{(l+1)}}$$

$$\frac{d^2u}{d\rho^2} = l(l+1)C\rho^{l-1} + l(l+1)\frac{D}{\rho^{(l+2)}}$$

but the second term blows up as $\rho \to 0$, so $D = 0$ and

$$u(\rho)|_{\rho\to 0} \sim C\rho^{l+1}$$

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

$$\frac{du}{d\rho} = (l+1)C\rho^l - l\frac{D}{\rho^{(l+1)}}$$

$$\frac{d^2u}{d\rho^2} = l(l+1)C\rho^{l-1} + l(l+1)\frac{D}{\rho^{(l+2)}}$$

but the second term blows up as $\rho \to 0$, so $D = 0$ and

$$u(\rho)|_{\rho\to 0} \sim C\rho^{l+1}$$

the full solution we seek, including the asymptotic portions is thus

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

$$\frac{du}{d\rho} = (l+1)C\rho^l - l\frac{D}{\rho^{(l+1)}}$$

$$\frac{d^2u}{d\rho^2} = l(l+1)C\rho^{l-1} + l(l+1)\frac{D}{\rho^{(l+2)}}$$

but the second term blows up as $\rho \to 0$, so $D = 0$ and

$$u(\rho)|_{\rho\to 0} \sim C\rho^{l+1}$$

the full solution we seek, including the asymptotic portions is thus

$$u(\rho) = \rho^{l+1}v(\rho)e^{-\rho}$$

Hydrogen atom (cont.)

for $\rho \to 0$ the solution must satisfy

$$\frac{d^2u}{d\rho^2} \approx \frac{l(l+1)}{\rho^2}u$$

this has a solution

$$u(\rho) = C\rho^{l+1} + \frac{D}{\rho^l}$$

this solution can be shown to satisfy the equation

$$\frac{du}{d\rho} = (l+1)C\rho^l - l\frac{D}{\rho^{(l+1)}}$$

$$\frac{d^2u}{d\rho^2} = l(l+1)C\rho^{l-1} + l(l+1)\frac{D}{\rho^{(l+2)}}$$

but the second term blows up as $\rho \to 0$, so $D = 0$ and

$$u(\rho)|_{\rho\to 0} \sim C\rho^{l+1}$$

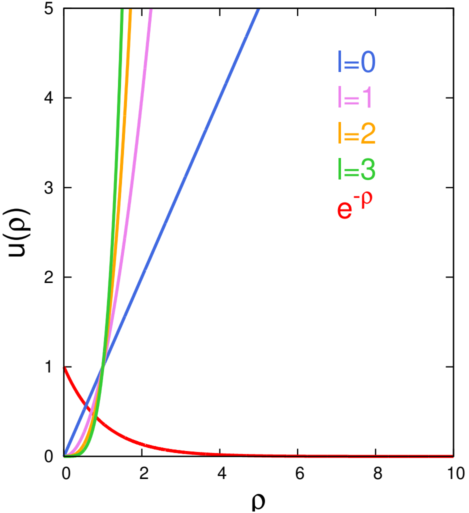
the full solution we seek, including the asymptotic portions is thus

$$u(\rho) = \rho^{l+1}v(\rho)e^{-\rho}$$

where $v(\rho)$ is a polynomial in ρ

Asymptotic behavior

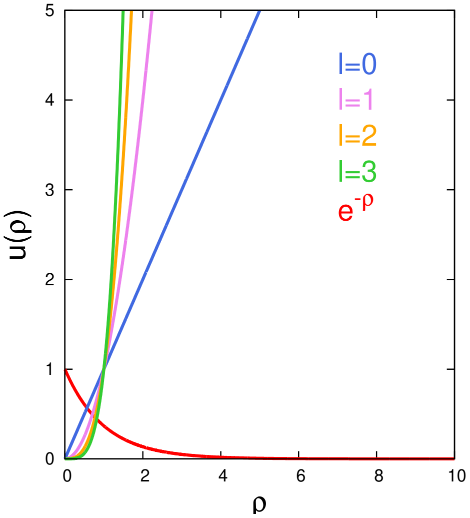
$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$



Asymptotic behavior

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

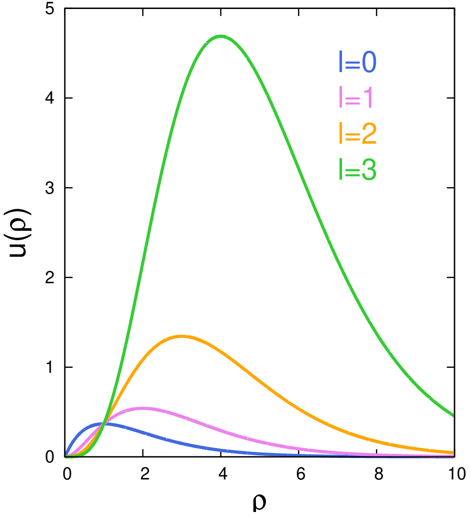
The exponential term serves to limit the asymptotic behavior of the wavefunction



Asymptotic behavior

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

The exponential term serves to limit the asymptotic behavior of the wavefunction

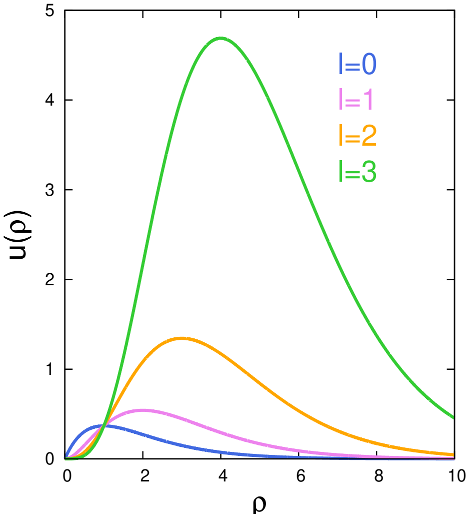


Asymptotic behavior

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

The exponential term serves to limit the asymptotic behavior of the wavefunction

It remains only to determine the polynomial "wavy part" of the solution, $v(\rho)$. This is done in the same way as was the analytical solution of the harmonic oscillator.



The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{d^2 u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} =$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^{l} e^{-\rho} v$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho}$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho \frac{dv}{d\rho}\right]$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} =$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} = l\rho^{l-1} e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho \frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} = l\rho^{l-1} e^{-\rho} \left[(l+1-\rho)v + \rho \frac{dv}{d\rho}\right] - \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho \frac{dv}{d\rho}\right]$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho \frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} = l\rho^{l-1} e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right] - \rho^l e^{-\rho}\left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$
$$+ \rho^l e^{-\rho}\left[-v + (l+1-\rho)\frac{dv}{d\rho} + \frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} = l\rho^{l-1} e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right] - \rho^l e^{-\rho}\left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$+ \rho^l e^{-\rho}\left[-v + (l+1-\rho)\frac{dv}{d\rho} + \frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$= \rho^l e^{-\rho} \left\{ \qquad \right\}$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} = l\rho^{l-1} e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right] - \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$+ \rho^l e^{-\rho} \left[-v + (l+1-\rho)\frac{dv}{d\rho} + \frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$= \rho^l e^{-\rho} \left\{\rho\frac{d^2v}{d\rho^2} \qquad \right\}$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$\frac{d^2u}{d\rho^2} = l\rho^{l-1} e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right] - \rho^l e^{-\rho} \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$+ \rho^l e^{-\rho} \left[-v + (l+1-\rho)\frac{dv}{d\rho} + \frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$= \rho^l e^{-\rho} \left\{\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} \qquad\qquad\qquad\qquad \right\}$$

The wavy part

The polynomial can be determined by substituting this solution into the original Schrödinger equation for $u(\rho)$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\frac{du}{d\rho} = (l+1)\rho^l e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho} = \rho^l e^{-\rho}\left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]$$

$$\begin{aligned}\frac{d^2u}{d\rho^2} &= l\rho^{l-1} e^{-\rho}\left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right] - \rho^l e^{-\rho}\left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right] \\ &\quad + \rho^l e^{-\rho}\left[-v + (l+1-\rho)\frac{dv}{d\rho} + \frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right] \\ &= \rho^l e^{-\rho}\left\{\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho}\right] v\right\}\end{aligned}$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2 u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho}\right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] \rho^{l+1} e^{-\rho} v$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2 u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2 u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u = \frac{d^2 u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] \rho^{l+1} e^{-\rho} v$$

$$= \rho^l e^{-\rho} \left\{ \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] \rho^{l+1} e^{-\rho} v$$

$$= \rho^l e^{-\rho} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v - \left[\rho - \rho_0 + \frac{l(l+1)}{\rho} \right] v \right\}$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2u}{d\rho^2} = \rho^l e^{-\rho}\left\{\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho}\right]v\right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]\rho^{l+1}e^{-\rho}v$$

$$= \cancel{\rho^l e^{-\rho}}\left\{\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho}\right]v - \left[\rho - \rho_0 + \frac{l(l+1)}{\rho}\right]v\right\}$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1}e^{-\rho}v(\rho), \quad \frac{d^2u}{d\rho^2} = \rho^l e^{-\rho}\left\{\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho}\right]v\right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]\rho^{l+1}e^{-\rho}v$$

$$= \cancel{\rho^l e^{-\rho}}\left\{\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho}\right]v - \left[\rho - \rho_0 + \frac{l(l+1)}{\rho}\right]v\right\}$$

$$0 = \rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho}$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2 u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2 u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u = \frac{d^2 u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] \rho^{l+1} e^{-\rho} v$$

$$= \cancel{\rho^l e^{-\rho}} \left\{ \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\cancel{\rho} - 2(l+1) + \cancel{\frac{l(l+1)}{\rho}} \right] v - \left[\cancel{\rho} - \rho_0 + \cancel{\frac{l(l+1)}{\rho}} \right] v \right\}$$

$$0 = \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho}$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] \rho^{l+1} e^{-\rho} v$$

$$= \cancel{\rho^l e^{-\rho}} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\cancel{\rho} - 2(l+1) + \cancel{\frac{l(l+1)}{\rho}} \right] v - \left[\cancel{\rho} - \rho_0 + \cancel{\frac{l(l+1)}{\rho}} \right] v \right\}$$

$$0 = \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(l+1)]v$$

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2 u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2 u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u = \frac{d^2 u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] \rho^{l+1} e^{-\rho} v$$

$$= \cancel{\rho^l e^{-\rho}} \left\{ \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\cancel{\rho} - 2(l+1) + \cancel{\frac{l(l+1)}{\rho}} \right] v - \left[\cancel{\rho} - \rho_0 + \cancel{\frac{l(l+1)}{\rho}} \right] v \right\}$$

$$0 = \rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(l+1)] v$$

we will solve this in the same way as the harmonic oscillator, assuming that $v(\rho)$ is an infinite polynomial in ρ

The Schrödinger equation for $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \frac{d^2u}{d\rho^2} = \rho^l e^{-\rho} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\rho - 2(l+1) + \frac{l(l+1)}{\rho} \right] v \right\}$$

Substituting into the Schrödinger equation for $u(\rho)$

$$0 = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u = \frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] \rho^{l+1} e^{-\rho} v$$

$$= \cancel{\rho^l e^{-\rho}} \left\{ \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + \left[\cancel{\rho} - 2(l+1) + \cancel{\frac{l(l+1)}{\rho}} \right] v - \left[\cancel{\rho} - \rho_0 + \cancel{\frac{l(l+1)}{\rho}} \right] v \right\}$$

$$0 = \rho \frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(l+1)]v$$

we will solve this in the same way as the harmonic oscillator, assuming that $v(\rho)$ is an infinite polynomial in ρ

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$$