The quantum harmonic oscillator

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- Classical harmonic oscillator

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- Factoring the Hamiltonian

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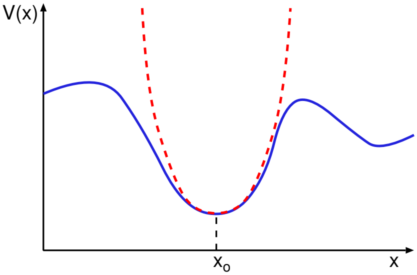

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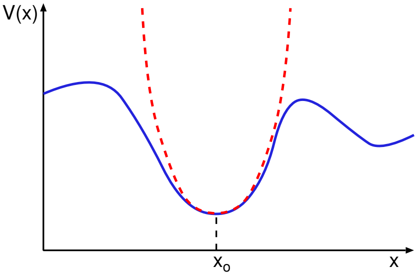

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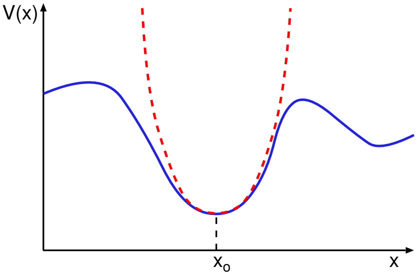

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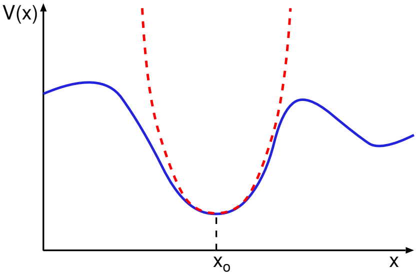

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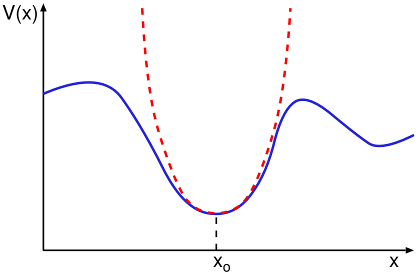

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Harmonic oscillator - algebraic solution

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- Generating new solutions recursively

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Solving the Schrödinger equation

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If ψ satisfies the Schrödinger equation with energy E, then what about $a_+\psi$?

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$a_+\psi$ also satisfies the Schrödinger equation, but with "energy" $E + \hbar\omega$!

Harmonic oscillator solutions

Just as $a_+\psi$ is a solution of the harmonic oscillator Hamiltonian

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Raising and lowering operators

The operators $a_\pm$ are called "raising" and "lowering" operators, respectively, and provide a recursive solution to the harmonic oscillator potential.

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